Topic: MOVING AVERAGE MODELS

A MOVING AVERAGE MODEL IS SIMPLY A LINEAR COMBINATION OF WHITE NOISE PROCESSES, SO THAT Y_t DEPENDS ON THE CURRENT AND PREVIOUS VALUES OF A WHITE NOISE DISTURBANCE TERM.

THEN

$$\boxed{Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q}}$$

IS A q^{th} ORDER MOVING AVERAGE MODEL, DENOTED AS MA (q)

THIS MODEL CAN BE ALSO EXPRESSED AS:

$$Y_t = \mu + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \varepsilon_t$$

→ PLEASE NOTE THAT ε_t IS A WHITE NOISE PROCESS WITH $E(\varepsilon_t) = 0$ AND $Var(\varepsilon_t) = \sigma^2$

AUTOREGRESSIVE MODELS

AN AUTOREGRESSIVE MODEL IS ONE WHERE THE CURRENT VALUE OF A VARIABLE, Y, DEPENDS UPON ONLY THE VALUES THAT THE VARIABLE TOOK IN PREVIOUS PERIODS PLUS AN ERROR TERM.

AN AUTOREGRESSIVE MODEL OF ORDER P, DENOTED AS AR(p), CAN BE EXPRESSED AS:

$$\boxed{Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \varepsilon_t}$$

where ε_t IS A WHITE NOISE DISTURBANCE TERM.

THE MODEL CAN BE ALSO EXPRESSED AS:

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t$$

OR USING THE LAG OPERATOR, AS

$$Y_t = \mu + \sum_{i=1}^{p} \phi_i L^i y_t + \varepsilon_i$$

ARMA MODELS

BY COMBINING THE AR(p) AND MA(q) MODELS, AN ARMA (p, q) MODEL IS OBTAINED.
SUCH A MODEL STATES THAT THE CURRENT VALUE OF A SERIES y DEPENDS LINEARLY ON ITS OWN PREVIOUS VALUES PLUS A COMBINATION OF CURRENT AND PREVIOUS VALUES OF A WHITE NOISE ERROR TERM. ARMA (p, q) IS WRITTEN AS:

$$\boxed{Y_t = \mu + \phi_1 Y_{t-1} + \dots + \phi_p Y_{t-p} + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q} + \varepsilon_t}$$

examples:

$AR(1) \rightarrow Y_t = \mu + \phi_1 Y_{t-1} + \varepsilon_t$

$MA(1) \rightarrow Y_t = \mu + \theta \varepsilon_{t-1} + \varepsilon_t$

$ARMA(1, 1) \rightarrow Y_t = \mu + \phi_1 Y_{t-1} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

Characteristics

THE DEFINING CHARACTERISTICS OF AR, MA AND ARMA PROCESSES CAN BE SUMMARISED AS:

	ACF	PACF
AR MODEL	A GEOMETRICALLY DECAYING ACF	A NUMBER OF NON-ZERO POINTS (spikes) = AR ORDER
MA MODEL	A NUMBER OF NON-ZERO POINTS (spikes) = MA ORDER	A GEOMETRICALLY DECAYING PACF
ARMA MODEL	A GEOMETRICALLY DECAYING ACF	A GEOMETRICALLY DECAYING PACF

Problem 4.3

ASSUME THAT SAMPLE SIZE $n = 400$ AND

- AUTOCORRELATION FUNCTION AT LAGS $1, \dots, 12$ IS

$-0.24, 0.32, -0.15, 0.12, 0.001, 0.05, 0.01, 0.011, 0.009, 0.04 \dots$

- PARTIAL AUTOCORRELATION FUNCTION IS

$0.4, 0.2, -0.05, 0.012, 0.04, 0.015, 0.011, 0.031, 0.0019, \dots$

i) TEST AT 5% SIGNIFICANCE LEVEL THAT THIS TIME SERIES IS A WHITE NOISE

To answer this question we use the ACF values

$\downarrow$

$H_0: \rho_K = 0$ where $K = 1, 2, \dots, 12$

$H_1: \rho_K \neq 0$

SO WE WANT TO TEST WHETHER THE CORRELATION COEFFICIENT AT LAG K IS SIGNIFICANT OR NOT.

- TEST STATISTIC:

$$t = \sqrt{n}\,\hat{\rho}_K \sim N(0,1)$$

- REJECTION RULE

→ CORRELATION AT LAG K IS SIGNIFICANT, IF
$|\hat{\rho}_K| > \frac{2}{\sqrt{n}}$ where n = obs.

→ CORRELATION AT LAG K IS NOT SIGNIFICANT,
IF $|\hat{\rho}_K| \leq \frac{2}{\sqrt{n}}$

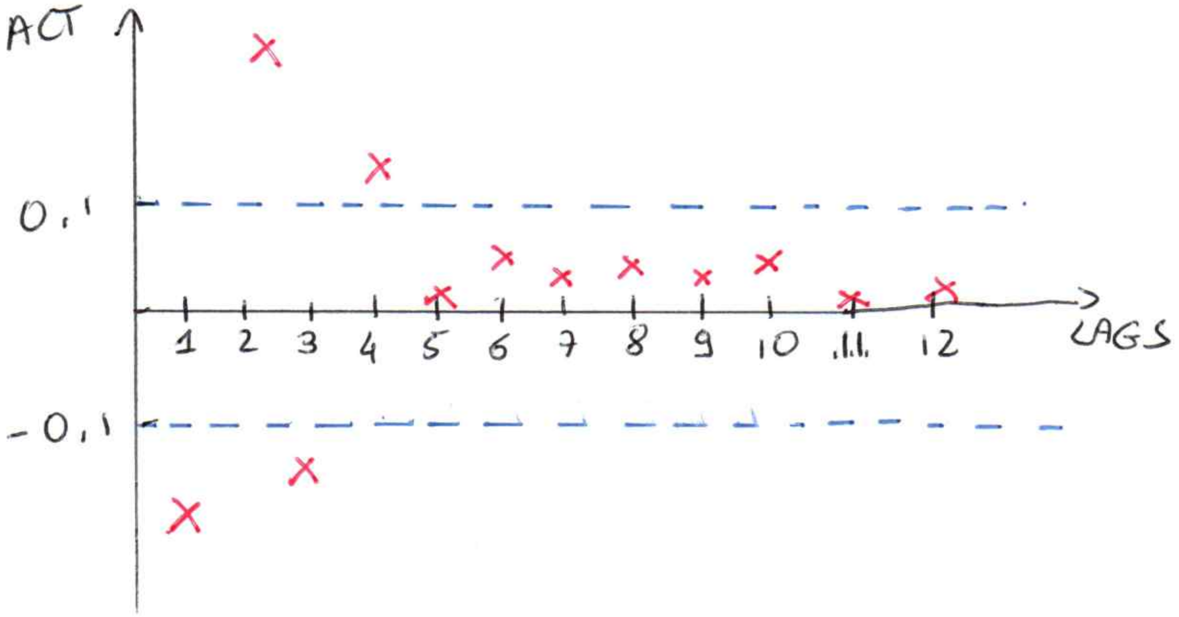
WE HAVE $n = 400$, SO:

$$\frac{2}{\sqrt{n}} = \frac{2}{\sqrt{400}} = 0,1$$

THEREFORE WE FAIL TO REJECT H_0, IF THE CORRELATION COEFFICIENT HAS A VALUE IN THE INTERVAL:

$$\left[-\frac{2}{\sqrt{n}} ; \frac{2}{\sqrt{n}}\right] = [-0.1 ; 0.1]$$

• GRAPHICALLY



THE LAST SIGNIFICANT LAG IS LAG 4

THE GRAPH OF ACF SHOWS SIGNIFICANT CORRELATION AT LAGS 1, 2, 3, 4.

THEREFORE, THE TIME SERIES IS NOT A WHITE NOISE.

ii) WHAT p WOULD YOU USE FITTING AR(p) MODEL TO THE DATA?

To select the order p of an AR model, we use PACF.

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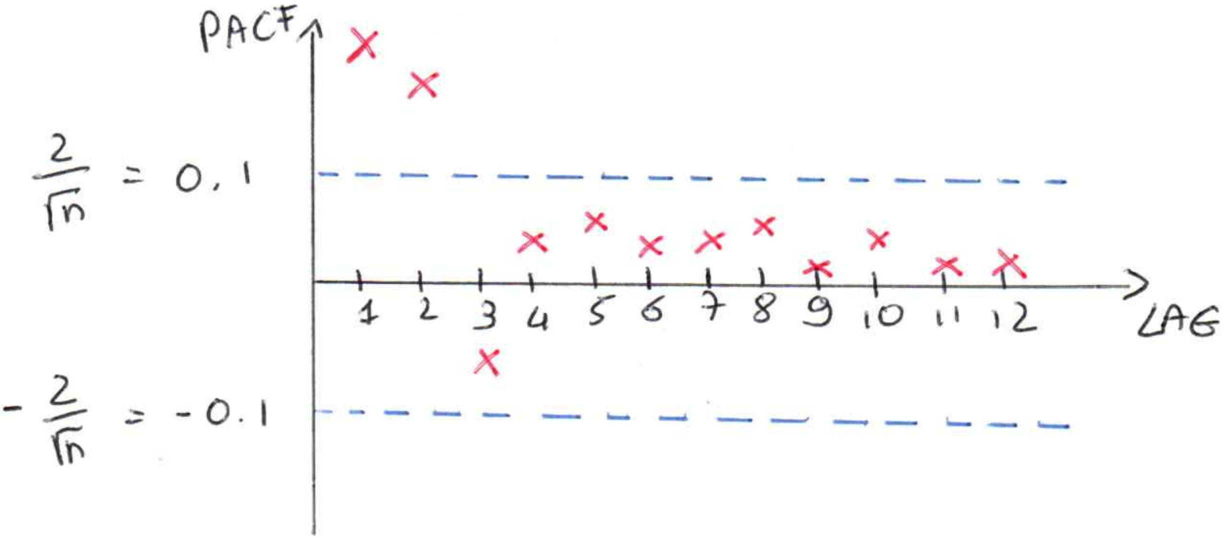
WE SELECT THE p AS THE LARGEST LAG AT WHICH THE PACF IS SIGNIFICANT.

THE RULE TO USE IS THE SAME AS IN i)



THE GRAPH SHOWS THAT THE LAST SIGNIFICANT PACF IS AT LAG 2

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THE DATA FIT AN $\boxed{\text{AR (2) MODEL}}$

$$AR(2): \quad X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

iii) WHAT q WOULD YOU USE FITTING MA(q) MODEL TO THE DATA ?

To select the order q of a MA model, we use ACF

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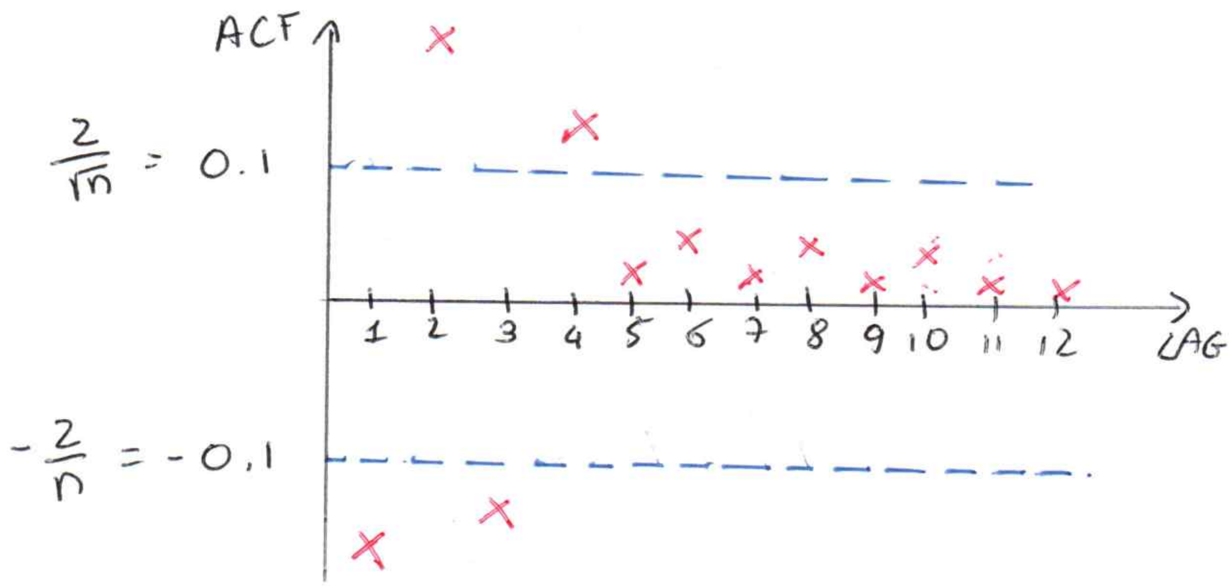
WE SELECT THE q AT THE LARGEST LAG AT WHICH THE ACF IS SIGNIFICANT



THE GRAPH SHOWS THAT THE LAST SIGNIFICANT ACF IS AT LAG 4

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THE DATA FIT AN $\boxed{\text{MA}(4) \text{ MODEL}}$

MA(4): $X_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \theta_3 \varepsilon_{t-3} + \theta_4 \varepsilon_{t-4}$

iv) WHICH MODEL (AR OR MA) WOULD YOU APPLY?

WE CAN FIT TO THE DATA EITHER
AR(2) MODEL OR MA(4) MODEL

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WE SELECT AR(2) MODEL BECAUSE IT HAS
A SMALLER NUMBER OF PARAMETERS

Problem 4.2

FIND THE MEAN FOR THE FOLLOWING AR PROCESSES

a) $X_t = 40 + 0.5\, X_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim WN(0, \sigma^2)$

$\Downarrow$

THIS IS A STATIONARY AR(1) MODEL

$$X_t = \mu + \phi\, X_{t-1} + \varepsilon_t$$

where $|\phi| < 1$

THE EXPECTED VALUE IS:

$$E(X_t) = E(a + \phi\, X_{t-1} + \varepsilon_t)$$

$$= \underbrace{a} + \phi \underbrace{E(X_{t-1})} + \underbrace{E(\varepsilon_t)} =$$

THE EXP. VALUE OF A CONSTANT IS THE CONSTANT ITSELF
$E(\mu) = \mu$

THIS IS A STATIONARY PROCESS, SO
$E(X_t) = E(X_{t-1}) = \mu_x$

ε_t IS A WHITE NOISE SO
$E(\varepsilon_t) = 0$

$$E(X_t) = a + \phi\, E(X_{t-1})$$

THEREFORE WE HAVE

$$\mu_x = a + \phi\, \mu_x$$

$$\mu_x - \phi\, \mu_x = a$$

$$(1 - \phi)\, \mu_x = a \iff \mu_x = \frac{a}{1 - \phi}$$

IN OUR CASE $a = 40$ AND $\phi = 0.5$

SO $\mu_x = \dfrac{a}{1-\phi} = \dfrac{40}{1-0.5} = 80$

b) $X_t = 20 + 0.2\, X_{t-1} + 0.6\, X_{t-2} + \varepsilon_t$

where $\varepsilon_t \sim WN(0, \delta^2)$

THIS IS A STATIONARY AR(2) MODEL

$$X_t = a + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$$

$$\to E(X_t) = E(a + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t)$$

$$E(X_t) = a + \phi_1 E(X_{t-1}) + \phi_2 E(X_{t-2}) + \underbrace{E(\varepsilon_t)}_{=0}$$

IT IS A STATIONARY PROCESS

SO $E(X_t) = E(X_{t-1}) = E(X_{t-2}) = \mu$

THEREFORE

$$\mu_x = a + \phi_1 \mu_x + \phi_2 \mu_x + 0$$

$$(1 - \phi_1 - \phi_2)\mu_x = a \iff \mu_x = \frac{a}{1-\phi_1-\phi_2}$$

IN OUR CASE : $a = 20$, $\phi_1 = 0.2$ AND $\phi_2 = 0.6$

SO $\mu_x = \dfrac{a}{1-\phi_1-\phi_2} = \dfrac{20}{1-0.2-0.6} = 100$