Vectors & Matrices

Solutions to Problem Sheet 4

1. (i) Let $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 6\mathbf{k}$ be the position vector of the centre of the sphere S. The set of points on the sphere S is given by the endpoints of all position vectors $\mathbf{p}$ such that the length of the vector $\mathbf{p} - \mathbf{c}$ is given by

$$|\mathbf{p} - \mathbf{c}| = 6\sqrt{3}\ .$$

Squaring both sides of this equation, we get $|\mathbf{p}-\mathbf{c}|^2 = (6\sqrt{3})^2 = 108$. If we take the coordinates of the position vectors $\mathbf{p}$ to be $\mathbf{p} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we derive the Cartesian equation

$$|\mathbf{p} - \mathbf{c}|^2 = (x-3)^2 + (y+1)^2 + (z-6)^2 = 108\ . \tag{1}$$

(ii) Any point on C is expressible as $(2\cos\theta + 6\sin\theta + 7\,,\ 8\cos\theta - 3\,,\ -2\cos\theta + 6\sin\theta + 2)$, for some $\theta \in \mathbb{R}$. To show that any such point lies on S, it suffices to substitute each coordinate of the point into our Cartesian equation (1):

$$\begin{aligned}
(x-3)^2 + (y+1)^2 + (z-6)^2 &= ((2\cos\theta + 6\sin\theta + 7) - 3)^2 + ((8\cos\theta - 3) + 1)^2 \\
&\quad + ((-2\cos\theta + 6\sin\theta + 2) - 6)^2 \\
&= (2\cos\theta + 6\sin\theta + 4)^2 + (8\cos\theta - 2)^2 + (-2\cos\theta + 6\sin\theta - 4)^2 \\
&= (2\cos\theta + 6\sin\theta)^2 + (8\cos\theta)^2 + (-2\cos\theta + 6\sin\theta)^2 \\
&\quad + 2[4(2\cos\theta + 6\sin\theta) - 2(8\cos\theta) - 4(-2\cos\theta + 6\sin\theta)] \\
&\quad + (4^2 + (-2)^2 + (-4)^2) \\
&= (4 + 64 + 4)\cos^2\theta + (36 + 36)\sin\theta + (12 - 12)\cos\theta\sin\theta \\
&\quad + 2(8 - 16 + 8)\cos\theta + 2(24 - 24)\sin\theta + (16 + 4 + 16) \\
&= 72(\cos^2\theta + \sin^2\theta) + 36 \\
&= 108\ .
\end{aligned}$$

Since the left-hand expression of the Cartesian equation of S evaluates to 108 for each point in C, C is entirely contained within the sphere S.

(iii) If C is a circle centred at $\mathbf{c} = 7\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, then every position vector $\mathbf{p}$ with endpoint in C should satisfy the equation $|\mathbf{p} - \mathbf{c}| = r$, for some fixed value $r \geq 0$. This r would then be the radius of the circle. Take some value $\theta \in [0, 2\pi)$,

$$\mathbf{p} = \begin{pmatrix} 2\cos\theta + 6\sin\theta + 7 \\ 8\cos\theta - 3 \\ -2\cos\theta + 6\sin\theta + 2 \end{pmatrix}$$

is a position vector satisfying $|\mathbf{p} - \mathbf{c}| = r$. We therefore compute

$$\begin{aligned}
|\mathbf{p} - \mathbf{c}|^2 &= \left| \begin{pmatrix} (2\cos\theta + 6\sin\theta + 7) - 7 \\ (8\cos\theta - 3) - 3 \\ (-2\cos\theta + 6\sin\theta + 2) - 2 \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 2\cos\theta + 6\sin\theta \\ 8\cos\theta \\ -2\cos\theta + 6\sin\theta \end{pmatrix} \right| \\
&= (2\cos\theta + 6\sin\theta)^2 + (8\cos\theta)^2 + (-2\cos\theta + 6\sin\theta)^2 \\
&= (4 + 64 + 4)\cos\theta + (36 + 36)\sin\theta + (12 - 12)\cos\theta\sin\theta \\
&= 72(\cos^2\theta + \sin^2\theta) \\
&= 72\ .
\end{aligned}$$

Hence, $r = |\mathbf{p} - \mathbf{c}| = \sqrt{72} = 6\sqrt{2}$.

(iv) We know that the centre of the sphere S is the point $R = (3, -1, 6)$. By Proposition 3.1.6, we have

$$\overrightarrow{RP} = \begin{pmatrix} 5 - 3 \\ 9 - (-1) \\ 8 - 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ 2 \end{pmatrix} .$$

The vector $\overrightarrow{RP}$ is pointing radially outward from the centre towards the surface point P. By a geometric argument, this radial vector is orthogonal to every vector along the tangent plane at P. Therefore, using the results from Section 5.3, the tangent plane is given by the equation $\mathbf{r} \cdot \overrightarrow{RP} = d$, where

$$d = \overrightarrow{OP} \cdot \overrightarrow{RP} = \begin{pmatrix} 5 \\ 9 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 10 \\ 2 \end{pmatrix} = 10 + 90 + 16 = 116 \ .$$

If we take $r = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then we can derive the Cartesian form of this equation:

$$2x + 10y + 2z = 116 \ ,$$

or cancelling out the common factor of two:

$$x + 5y + z = 58 \ .$$

2. (i) It's clear that every point on the *Red Line* can be identified as the endpoint of position vectors $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$, for any $\lambda \in \mathbb{R}$, where $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$ is the position vector of *Portway* station and $\mathbf{u} = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$ is the direction of travel. If we take $\mathbf{x} = 5\mathbf{i} + 8\mathbf{k}$ as being the position vector of the housing development, then the vector $\mathbf{u} \times (\mathbf{x} - \mathbf{p})$ is given by

$$\begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \cdot 13 - 6 \cdot 5 \\ 6 \cdot 2 - 1 \cdot 13 \\ 1 \cdot 5 - 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} -17 \\ -1 \\ 3 \end{pmatrix}$$

Thus, from the formula in Section 5.8, the distance of the closest approach of the *Red Line* to $(5, 0, 8)$ is equal to

$$\frac{|\mathbf{u} \times (\mathbf{x} - \mathbf{p})|}{|\mathbf{u}|} = \frac{\sqrt{(-17)^2 + (-1)^2 + 3^2}}{\sqrt{1^2 + 1^2 + 6^2}} = \sqrt{\frac{299}{38}} \approx 2.81 \ .$$

(ii) The *Blue Line* can be identified as the endpoints of position vectors $\mathbf{b} = \mathbf{q} + \mu\mathbf{v}$, for any $\mu \in \mathbb{R}$, where $\mathbf{q} = 2\mathbf{i} - 8\mathbf{j} + \mathbf{k}$ is the position vector of *Queen's Road* station, and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ is the direction of travel. We have

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \cdot (-3) - 6 \cdot (-1) \\ 6 \cdot 2 - 1 \cdot (-3) \\ 1 \cdot (-1) - 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{q} - \mathbf{p} = \begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix},$$

hence, by the formulation of the distance between two lines given in Section 5.9, the minimal distance between the *Red* and *Blue Lines* is

$$\frac{|(\mathbf{q} - \mathbf{p}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|(-1) \cdot 3 + (-3) \cdot 15 + 6 \cdot (-3)|}{\sqrt{3^2 + 15^2 + (-3)^2}} = \frac{|-66|}{\sqrt{243}} = \frac{22\sqrt{3}}{9} \approx 4.23\,.$$

(iii) Let R be the point on the *Red Line* that is the site of the new tunnel opening. Similarly, let B be the point at which the new tunnel opens onto the *Blue Line*. Since we can only move through the tunnels, the distance travelled between *Portway* and *Queen's Road* is given by the sum of the distance between *Portway* and point R, the length of the new tunnel itself, and the distance between point B and *Queen's Road* station.

Labelling P as the point at which *Portway* station lies, and Q as the site of *Queen's Road* station, we are aiming to evaluate the sum $|\overrightarrow{PR}| + |\overrightarrow{RB}| + |\overrightarrow{BQ}|$. The value $|\overrightarrow{RB}|$ is the length of the tunnel, known to us from part (ii). To compute the values of the other two terms, we will need the value

$$\alpha = \frac{(\mathbf{q} - \mathbf{p}) \cdot (\mathbf{u} \times \mathbf{v})}{|\mathbf{u} \times \mathbf{v}|^2} = \frac{-66}{243} = \frac{-22}{81}\,.$$

Per Section 5.9 in the lecture notes, $\alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{q} + \mu\mathbf{v} - \mathbf{p} - \lambda\mathbf{u}$, where $\lambda\mathbf{u}$ represents the vector connecting the point P and the point on the *Red Line* closest to the *Blue Line* (and similarly for vector $\mu\mathbf{v}$ and Q along the *Blue Line*). By our definition of R and B, this means that $|\lambda\mathbf{u}| = |\overrightarrow{PR}|$ and $|\mu\mathbf{v}| = |\overrightarrow{BQ}|$.

All that remains is to find the values of λ and μ. Given that the equation $\alpha(\mathbf{u} \times \mathbf{v}) = \mathbf{q} + \mu\mathbf{v} - \mathbf{p} - \lambda\mathbf{u}$ holds in all three dimensions (and that we have the coordinates of vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$, plus a value for α), we can derive the following system of equations

$$\begin{cases} 3\alpha = 2 + 2\mu - 3 - \lambda \\ 15\alpha = -8 - \mu - (-5) - \lambda \\ -3\alpha = 1 - 3\mu - (-5) - 6\lambda \ , \end{cases}$$

or equivalently

$$\begin{cases} 2\mu - \lambda = 3\alpha + 1 = \frac{5}{27} \\ -\mu - \lambda = 15\alpha + 3 = \frac{-29}{27} \\ -3\mu - 6\lambda = -3\alpha - 6 = \frac{-140}{27} \ . \end{cases}$$

These are three equations in two unknown variables, λ and μ. We can multiply the second equation by 2 and add it to the first to get $-3\lambda = \frac{-53}{27}$, giving us $\lambda = \frac{53}{81}$. Similarly, substituting this value back into second equation gives us $\mu = \frac{29}{27} - \lambda = \frac{34}{81}$. We can substitute both values into the last equation to show that it is also satisfied.

Combining these values with the analysis form earlier, we have that the total distance that needs to be travelled is given by

$$\begin{aligned} |\overrightarrow{PR}| + |\overrightarrow{RB}| + |\overrightarrow{BQ}| &= |\lambda \mathbf{u}| + |\overrightarrow{RB}| + |\mu \mathbf{v}| \\ &= |\lambda||\mathbf{u}| + |\overrightarrow{RB}| + |\mu||\mathbf{v}| \\ &= \frac{53}{81}\sqrt{38} + \frac{22\sqrt{3}}{9} + \frac{34}{81}\sqrt{14} \\ &\approx 7.02 \ . \end{aligned}$$

3. (i) Let $f(x) = ax^2 + bx + c$ represent any quadratic in our set. The value of such an expression at $x = -2$ is given by

$$f(-2) = a(-2)^2 + b(-2) + c = 4a - 2b + c \ .$$

The property given in the question requires that $f(-2) = 31$, and so we can use the above formulation to express this property in terms of the following constraint on the coefficients of the quadratic

$$4a - 2b + c = 31 \ .$$

This equation is in the same form as the Cartesian equation of a plane in three dimensional space. Therefore, if we treat the coefficients of quadratic functions as a three dimensional space, the condition that $f(-2) = 31$ can be viewed geometrically as a plane.

(ii) Again, we represent any quadratic in our set as $f(x) = ax^2 + bx + c$. The condition that f has remainder $x + 5$ after division by $x^2 - 3x + 4$ can be written as

$$f(x) = \lambda(x^2 - 3x + 4) + (x + 5) \ ,$$

for some $\lambda \in \mathbb{R}$. Equation the coefficients of the x^2, x and constant terms, we derive the system

$$\begin{cases} a = \lambda \\ b = 1 - 3\lambda \\ c = 5 + 4\lambda \end{cases} .$$

This system is equivalent to the parametrisation of a line in three dimensional space, and so we can view the condition that a quadratic have remainder $x+5$ after division by $x^2 - 3x + 4$ as being a line in the space of coefficients.

(iii) We have shown in part (i) that the constraint $f(-2) = 31$ gives a plane in the space of coefficients, and that the function f having a remainder of $x + 5$ after division by $x^2 - 3x + 4$ gives a line. The set of coefficients (a, b, c) that result in a quadratic satisfying both constraints is therefore given by the intersection of this plane and line.

To evaluate all points within the intersection, it suffices to find all values (a, b, c) that satisfy both equations. The parameterisation of the line gives formulations for values a, b and c in terms of some parameter $\lambda \in \mathbb{R}$. We can substitute these expressions into the equation of the plane, and get

$$\begin{aligned} 4a - 2b + c &= 4(\lambda) - 2(1 - 3\lambda) + (5 + 4\lambda) \\ &= 14\lambda + 3 \\ &= 31 \ . \end{aligned}$$

Solving for λ, we get $\lambda = 2$. Since this is the only value of λ resulting in coefficients (a, b, c) that solve both equations, there is only a single point on the intersection between the plane and the line, and thus only a single quadratic that satisfies both constraints.

To identify this quadratic, substitute this value of λ into the parameterisation for a, b and c and construct the quadratic with these coefficients.

$$\begin{cases} a = \lambda = 2 \\ b = 1 - 3\lambda = 1 - 3 \cdot 2 = -5 \\ c = 5 + 4\lambda = 5 + 4 \cdot 2 = 13 \; . \end{cases}$$

Hence, the only quadratic solving both constraints is $f(x) = 2x^2 - 5x + 13$. Indeed,

$$f(-2) = 2(-2)^2 - 5(-2) + 13 = 31 \; ,$$
$$f(x) = 2(x^2 - 3x + 4) + (x + 5) \; .$$