Vectors & Matrices

Solutions to Problem Sheet 3

1. For any real value $u \in \mathbb{R}$, there exists a value $\lambda \in \{-1, 1\}$ such that $|u| = \lambda u$.

   This means that for any vector $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, there exists a vector $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \in \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x, y, z \in \{-1, 1\} \right\}$,

   such that

$$\lambda \cdot \mathbf{u} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = |u_1| + |u_2| + |u_3| \ .$$

   Applying the Cauchy-Schwarz Inequality to $\lambda \cdot \mathbf{u}$, we get

$$\lambda \cdot \mathbf{u} \leq |\lambda||\mathbf{u}| \ ,$$

   and since $|\lambda| = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} = \sqrt{(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2} = \sqrt{3}$, this bound becomes

$$\lambda \cdot \mathbf{u} \leq \sqrt{3}|\mathbf{u}| \ ,$$

   giving us the result.

2. (i) The points on the line-segment L can be identified as the end points of the position vectors in the set $\{\lambda\mathbf{i} + (1-\lambda)\mathbf{j} : 0 \leq \lambda \leq 1\}$. Take the vector $\mathbf{v} = \lambda\mathbf{i} + (1-\lambda)\mathbf{j}$, for some $\lambda \in [0, 1]$. Applying the Triangle Inequality to vectors $\lambda\mathbf{i}$ and $(1-\lambda)\mathbf{j}$, we have

$$\begin{aligned} |\,\mathbf{v}\,| &= |\,\lambda\mathbf{i} + (1-\lambda)\mathbf{j}\,| \\ &\leq |\,\lambda\mathbf{i}\,| + |\,(1-\lambda)\mathbf{j}\,| \ . \end{aligned}$$

   In Problem Sheet 1 (Q5), we proved the absolute-homogeneity of the length operator, i.e. that for any vector $\mathbf{u}$ and scalar $\lambda \in \mathbb{R}$, $|\lambda\mathbf{u}| = |\lambda||\mathbf{u}|$. We also know that $|\,\mathbf{i}\,| = \sqrt{1^2 + 0^2 + 0^2} = 1$ and $|\,\mathbf{j}\,| = \sqrt{0^2 + 1^2 + 0^2} = 1$, and so

$$
\begin{aligned}
|\,\mathbf{v}\,| &\le |\,\lambda\mathbf{i}\,| + |\,(1-\lambda)\mathbf{j}\,| \\
&= |\,\lambda\,|\,|\,\mathbf{i}\,| + |\,1-\lambda\,|\,|\,\mathbf{j}\,| \\
&= \lambda\cdot 1 + (1-\lambda)\cdot 1 \qquad\qquad (\text{as } 0 \le \lambda \le 1) \\
&= 1\ .
\end{aligned}
$$

For any point on the line L, its distance from the origin is precisely given by the length of its position vector. Since we have just shown that such a length must be less than or equal to 1, we can see that 1 is the largest possible distance between a point on L and the origin.

(ii) Consider the points on the following four line segments:

- L_1, the line segment given in part (i);
- L_2, the segment L_1 reflected along the y-axis;
- L_3, the segment L_1 reflected along the x-axis;
- L_4, the segment L_1 reflected along the x and y-axes.

Together, the union of these four segments forms a square centred at the origin. Let A and B be any two points on this square. By Proposition 3.1.6, we can express the vector $\overrightarrow{AB}$ as $\overrightarrow{AO} + \overrightarrow{OB}$. We can apply the Triangle Inequality to this expression:

$$
|\overrightarrow{AB}| = |\overrightarrow{AO} + \overrightarrow{OB}| \le |\overrightarrow{AO}| + |\overrightarrow{OB}|\ . \tag{1}
$$

From part (i), the distance between any point on the line segment L and the origin O is at most 1. By the reflectional symmetry of the square around both axes, this same argument holds for all four line segments L_i. Since A and B each lie on one of these segments, $|\overrightarrow{AO}|$ and $|\overrightarrow{OB}|$ are both bounded above by 1. Applying these bounds to the inequality (1) above, we get

$$
|\overrightarrow{AB}| \le |\overrightarrow{AO}| + |\overrightarrow{OB}| \le 1 + 1 = 2\ .
$$

Hence, the distance between any two points on our square cannot be greater than 2. By Pythagoras's Theorem, the lengths of each side of this square are equal to $\sqrt{2}$. Since the distances between points in a square cannot change during rotation and translation, we see that the maximal distance between *any* square with side-lengths $\sqrt{2}$ is equal to 2.

Finally, for such a square, we can multiply all position vectors leading to points in the square by a factor of $\frac{\sqrt{2}}{2}$ to form a new, smaller square. The side-lengths of this new square would scale to $\frac{\sqrt{2}}{2} \cdot \sqrt{2} = \frac{2}{2} = 1$. The maximal distance between any two points scales down to $\frac{\sqrt{2}}{2} \cdot 2 = \sqrt{2}$, giving us the result.

(iii) Let S_1 and S_2 be two squares with side-lengths of 1 with a non-empty intersection. Let P be a point inside this intersection. For any two points A and B inside $S_1 \cup S_2$, we have

$$|\overrightarrow{AB}| = |\overrightarrow{AP} + \overrightarrow{PB}| \leq |\overrightarrow{AP}| + |\overrightarrow{PB}| \,. \tag{2}$$

As the point P lies in the intersection between the two squares, A and P must both lie inside the same square. Hence, by part (ii), $|\overrightarrow{AP}| \leq \sqrt{2}$. Similarly, $|\overrightarrow{PB}| \leq \sqrt{2}$. Applying these bounds to the inequality (2), we see that

$$|\overrightarrow{AB}| \leq |\overrightarrow{AP}| + |\overrightarrow{PB}| \leq \sqrt{2} + \sqrt{2} = 2\sqrt{2} \,.$$

3. (i) Let $\mathbf{p} = 2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ be the position vector with end point P. For any point $R = (x, y, z)$ on the surface of Π, we have $(\mathbf{r} - \mathbf{p}) \cdot \mathbf{n} = 0$, where $\mathbf{r}$ is the position vector with end point R. Hence, the equation

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n} \tag{3}$$

defines the plane Π. We know that $\mathbf{r} \cdot \mathbf{n} = 6x + y + 5z$ and $\mathbf{p} \cdot \mathbf{n} = 2 \cdot 6 + 2 \cdot 1 + (-2) \cdot 5 = 4$. Thus, the equation (3) reduces to the Cartesian form

$$6x + y + 5z = 4 \,.$$

(ii) By Proposition 5.4.1, for any vector $\mathbf{q}$, the minimal distance between $\mathbf{q}$ and the plane Π is given by

$$\frac{|\,\mathbf{q} \cdot \mathbf{n} - d\,|}{|\,\mathbf{n}\,|} \,,$$

where $d = \mathbf{p} \cdot \mathbf{n}$, which (per part (i)) we can determine to be equal to 4. We can express any point on the parabola C as the end point of position vector $\mathbf{q} = \lambda\mathbf{i} + \lambda^2\mathbf{j} + 4\mathbf{k}$, for some $\lambda \in \mathbb{R}$. This gives us

$$\mathbf{q} \cdot \mathbf{n} - d = 6\lambda + \lambda^2 + 5 \cdot 4 - 4 = \lambda^2 + 6\lambda + 16 \,.$$

This expression is a quadratic in λ. Completing the square, we get

$$\mathbf{q} \cdot \mathbf{n} - d = (\lambda + 3)^2 + 7 \ .$$

This formulation tells us that $\mathbf{q} \cdot \mathbf{n} - d$:

- Is always strictly positive;
- Has a minimum value of 7;
- Attains its minimum at $\lambda = -3$.

Therefore, $|\, \mathbf{q} \cdot \mathbf{n} - d \,|$ has a minimum value of 7 at $\lambda = -3$. Since $|\, \mathbf{n} \,|$ is a constant value that is not dependent on λ, we can equivalently state that

$$\frac{|\, \mathbf{q} \cdot \mathbf{n} - d \,|}{|\, \mathbf{n} \,|} = \frac{(\lambda + 3)^2 + 7}{\sqrt{62}}$$

has a minimum value of $\frac{7}{\sqrt{62}} \approx 0.89$, and that this value is the minimal distance between C and Π.

4. (i) This can be computed using the formulation in the definition of the vector product:

$$\mathbf{i} \times \mathbf{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 - 0 \cdot 1 \\ 0 \cdot 0 - 1 \cdot 0 \\ 1 \cdot 1 - 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k} \ .$$

This could also be partly answered by using Proposition 5.5.3. This result tells us that the vector $\mathbf{i} \times \mathbf{j}$ must be orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. By direct computation, we can find that $\mathbf{i} \cdot \mathbf{k} = 0$ and $\mathbf{j} \cdot \mathbf{k} = 0$, and so $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. In three dimensions, this means that $\mathbf{i} \times \mathbf{j}$ must be equal to $\lambda\mathbf{k}$ for some $\lambda \in \mathbb{R}$. Secondly, we have

$$|\, \mathbf{i} \times \mathbf{j} \,| = |\, \mathbf{i} \,||\, \mathbf{j} \,| \sin\theta \ ,$$

where θ is equal to the angle between $\mathbf{i}$ and $\mathbf{j}$. We have already shown that $\mathbf{i}$ and $\mathbf{j}$ are orthogonal, and so $\theta = \frac{\pi}{2}$. Thus, $|\, \mathbf{i} \times \mathbf{j} \,| = |\, \mathbf{i} \,||\, \mathbf{j} \,| = 1 \cdot 1 = 1$. Combining this with our result above, we see that

$$|\, \mathbf{i} \times \mathbf{j} \,| = |\, \lambda\mathbf{k} \,| = |\lambda||\, \mathbf{k} \,| = |\lambda| \cdot 1 = |\lambda| = 1 \ .$$

This tells us that λ must be equal to either 1 or -1. In fact, the vector product of two orthogonal vectors is always oriented in the direction that matches the orientation of the standard axis arrangement in three-dimensional space (a result commonly referred to as the 'right-hand rule', where the individual axes can be represented by the index finger, middle finger and thumb of the right hand).

(ii) To evaluate this vector, we can use the identity $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$ (for any vectors $\mathbf{u}$ and $\mathbf{v}$). In this case, since we have already found that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, we have

$$\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k} \, .$$

(iii) This is best done via direct computation:

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} (-1) \cdot 1 - 2 \cdot 4 \\ 2 \cdot 1 - 3 \cdot 1 \\ 3 \cdot 4 - (-1) \cdot 1 \end{pmatrix} = \begin{pmatrix} -9 \\ -1 \\ 13 \end{pmatrix}$$

(iv) Since both vectors in the product are equal, it is clear that the angle between them is equal to zero. Thus, by Proposition 5.5.3:

$$\left| \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \times \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \right| = \left| \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \right|^2 \sin(0) = 0 \, .$$

The resulting vector has length 0, and so it can only be the zero vector, giving us

$$\begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} \times \begin{pmatrix} 17 \\ 119 \\ -53 \end{pmatrix} = \mathbf{0} \, .$$

5. (i) To identify three distinct points in C, we select the following three values of the parameter θ:

- At $\theta = 0$, we have $(4, 1, 2) \in C$;
- At $\theta = \frac{\pi}{2}$, we have $(19, -4, 2) \in C$;
- At $\theta = \pi$, we have $(4, -1, -2) \in C$.

(Note: This is just a single example of three points in C, this exercise could be completed with *any* three points in C, as long as they are all distinct.)

If we take

$$\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \quad , \quad \mathbf{b} = \begin{pmatrix} 19 \\ -4 \\ 2 \end{pmatrix} \quad , \quad \mathbf{c} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$$

to be the position vectors leading to these three points, and denote the position vector leading to any point $R = (x, y, z)$ as $\mathbf{r}$, then the Cartesian equation of the plane containing the points is given by

$$\mathbf{r} \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) = \mathbf{a} \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})) \ .$$

We have:

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{pmatrix} 15 \\ -5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} (-5) \cdot (-4) - 0 \cdot (-2) \\ 0 \cdot 0 - 15 \cdot (-4) \\ 15 \cdot (-2) - (-5) \cdot 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 60 \\ -30 \end{pmatrix} \ .$$

Thus, the left side of our equation reduces to $\mathbf{r} \cdot (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = 20x + 60y - 30z$. Similarly, our right hand side evaluates to $\mathbf{a} \cdot (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = 4 \cdot 20 + 1 \cdot 60 + 2 \cdot (-30) = 80$. The Cartesian equation of this plane is therefore given by

$$20x + 60y - 30z = 80 \ ,$$

or equivalently, dividing all terms by the common factor of 10,

$$2x + 6y - 3z = 8 \ .$$

(ii) It is clear from the Cartesian equation derived in part (i) that the vector $\mathbf{n} = 2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ is orthogonal to the plane. For any vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, this equation is equivalent to $\mathbf{r} \cdot \mathbf{n} = d$, with $d = 8$.

By Proposition 5.4.1, if we take $\mathbf{q} = \mathbf{0}$ (since the zero vector is equal to the position vector pointing to the origin), the position vector of the point in the plane closest to the origin is given by

$$\mathbf{q} - \left(\frac{\mathbf{q} \cdot \mathbf{n} - d}{|\mathbf{n}|^2} \right) \mathbf{n} = \mathbf{0} - \left(\frac{\mathbf{0} \cdot \mathbf{n} - 8}{|\mathbf{n}|^2} \right) \mathbf{n} = \mathbf{0} - \left(\frac{0 - 8}{2^2 + 6^2 + (-3)^2} \right) \mathbf{n} = \frac{8}{49} \mathbf{n} \ ,$$

with the corresponding distance between this point and the plane given by

$$\frac{|\mathbf{q}\cdot\mathbf{n}-d|}{|\mathbf{n}|}=\frac{|\mathbf{0}\cdot\mathbf{n}-8|}{|\mathbf{n}|}=\frac{0-8}{\sqrt{49}}=\frac{8}{7}\ .$$

(iii) In order for a point $R=(x,y,z)$ to lie on the plane, it would need to satisfy the Cartesian equation derived in part (i). By the definition of the circle C, we know that all points that lie in C are of the form $(15\sin\theta+4,\ \cos\theta-4\sin\theta,\ 2\cos\theta+2\sin\theta)$, for some $\theta\in\mathbb{R}$.

Therefore, for any point in C, we can identify some value $\theta\in\mathbb{R}$ such that the coordinates (x,y,z) are given by

$$\begin{aligned}x&=15\sin\theta+4\\y&=\cos\theta-4\sin\theta\\z&=2\cos\theta+2\sin\theta\ .\end{aligned}$$

Substituting these values into the Cartesian equation, we get

$$\begin{aligned}2x+6y-3z&=2(15\sin\theta+4)+6(\cos\theta-4\sin\theta)-3(2\cos\theta+2\sin\theta)\\&=30\sin\theta+8+6\cos\theta-24\sin\theta-6\cos\theta-6\sin\theta\\&=8\ .\end{aligned}$$

This demonstrates that any such point on C would have coordinates satisfying the Cartesian equation of the plane, and would therefore lie on it. Since this can be applied to every point of C, it shows that the circle C is completely contained within the plane.

(iv) By part (iii), every point on the circle C lies within the plane $2x+6y-3z=8$. Using the contrapositive argument, any point that does not satisfy this Cartesian equation cannot lie on C. Substituting the coordinates of $(1,-1,1)$ into the left hand side of this equation, we get $2\cdot1+6\cdot(-1)-3\cdot1=-7$. Since this does not equal 8, we see that the point $(1,-1,1)$ does not lie on the plane, and is therefore not contained in C.