Lines through the origin and products of vectors

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Lines through the origin and example of sub-vector spaces

Let us consider the equation of the line l passing through the origin O and defined via the vector $\mathbf{u}$, i.e., $\mathbf{r} = \lambda\mathbf{u}$, for $\lambda \in \mathbb{R}$. This gives

$$V = \{\lambda\mathbf{u} : \lambda \in \mathbb{R}\}.$$

Proposition 4.2.1

For all $v, v_1, v_2 \in V$ and all $\alpha \in \mathbb{R}$,

$$v_1 + v_2 \in V,$$
$$\alpha v \in V.$$

Proposition 4.2.2

Let $\mathbf{i}$ and $\mathbf{j}$ be the standard vector in $\mathbb{R}^2$. The set,

$$V = \{x\mathbf{i} + y\mathbf{j} : \, x, y \in \mathbb{R}\},$$

is a sub-vector space of $\mathbb{R}^2$.

Scalar product

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors with $\overrightarrow{AB}$ representing $\mathbf{u}$ and $\overrightarrow{AC}$ representing $\mathbf{v}$, we define the *angle between* $\mathbf{u}$ *and* $\mathbf{v}$ to be the angle θ (in radians) between the line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with $0 \leq \theta \leq \pi$.

Definition 5.1.1

The *scalar product* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} |\mathbf{u}||\mathbf{v}| \cos\theta & \text{if } \mathbf{u} \neq \mathbf{0}, \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Definition 5.1.2

We say that $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem 5.1.3

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. Then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Remark 5.1.4

Proposition 5.1.5

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\alpha \in \mathbb{R}$ we have

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$,
3. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$,
4. $(\alpha \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha \mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$.

Interesting inequalities

Cauchy-Schwarz Inequality

Let **u** and **v** be two vectors in $\mathbb{R}^3$. The following inequality holds:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

Triangle inequality

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$. The following inequality holds:

$$|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

The equation of a plane

Distance from a point to a plane

Proposition 5.4.1

If the plane Π has equation $\mathbf{r} \cdot \mathbf{n} = d$, and the point Q has position vector $\mathbf{q}$, then the distance between Q and Π is

$$\frac{|\mathbf{q} \cdot \mathbf{n} - d|}{|\mathbf{n}|},$$

and the point on Π that is closest to Q has position vector

$$\mathbf{q} - \left(\frac{\mathbf{q} \cdot \mathbf{n} - d}{|\mathbf{n}|^2} \right) \mathbf{n}.$$