Lecture 9

19 February 2024 14:09

<u>Last time:</u>

Let R be a ring and I an ideal. We defined the quotient ring R/I where the elements are the cosets of I in R and operations

$$\left.\begin{aligned} [a]_I + [b]_I &:= [a+b]_I \\ [a]_I \cdot [b]_I &:= [a \cdot b]_I \end{aligned}\right\} \text{well defined}$$

<u>Thm</u>: R/I defined this way is a ring.

<u>Proof</u>: (A1)

$$\left([a]_I + [b]_I\right) + [c]_I \overset{\text{def of addition in } R/I}{=} [a+b]_I + [c]_I$$

$$= [(a+b)+c]_I$$

$$\overset{R \text{ satisfies (A0)}}{=} [a+(b+c)]_I$$

$$= [a]_I + [b+c]_I$$

$$\overset{\text{def of addition in } R/I}{=} [a]_I + \left([b]_I + [c]_I\right)$$

Similar for (M1), (D), (A4) (Exercise!)

(A2) The zero element of R/I is $[0]_I$.

This is equal to the ideal I.

(A3) The negative of $[a]_I$ is $[-a]_I$

$$-[a]_I = [-a]_I.$$

(M2) If R has an identity 1 then R/I also has an identity $[1]_I$.

(M3) If R is a division ring then R/I is also a division ring.

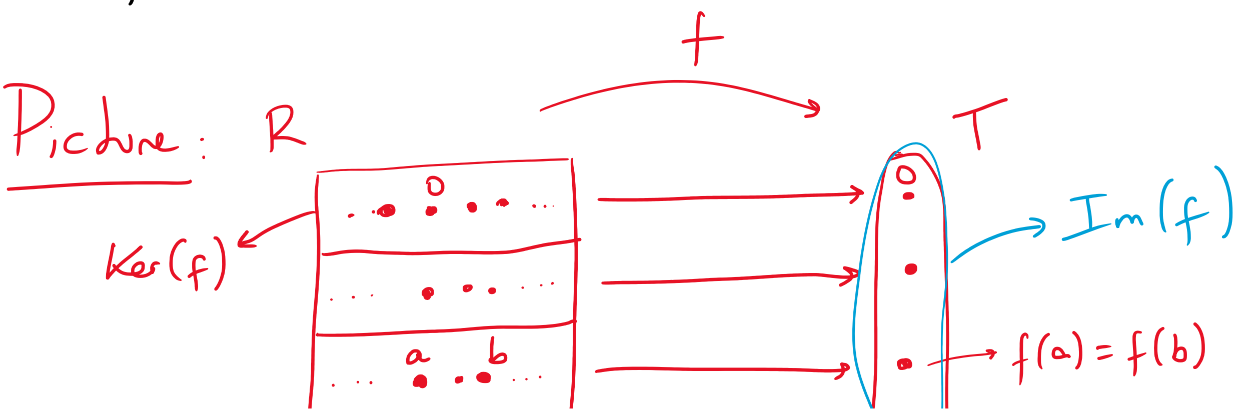
⚠ We saw an example where R did not have an identity but R/I did.

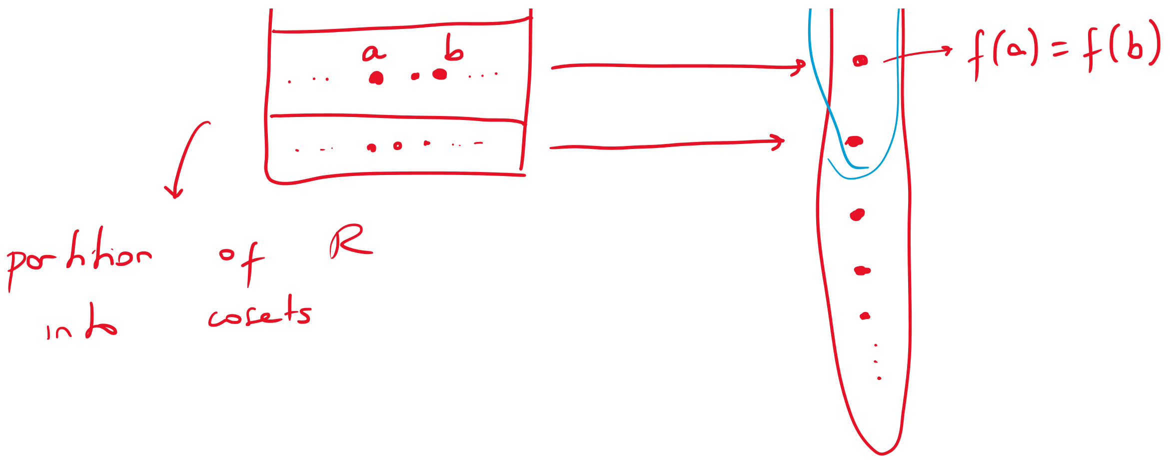
Isomorphism theorems:

Proposition: Let $f: R \longrightarrow T$ be a homomorphism of rings. Then

$$f(a) = f(b) \iff [a]_{Ker(f)} = [b]_{Ker(f)}.$$

Picture:





Proof:

$$
\begin{aligned}
f(a) = f(b) &\iff f(a) - f(b) = 0_T \\
&\iff f(a-b) = 0_T \\
&\iff a-b \in \mathrm{Ker}(f) \\
&\iff a \sim_{\mathrm{Ker}(f)} b \\
&\iff [a]_{\mathrm{Ker}(f)} = [b]_{\mathrm{Ker}(f)}
\end{aligned}
$$

$\blacksquare$

First Isomorphism Theorem:

Let $f : R \longrightarrow T$ be a ring homomorphism.

Then there is an isomorphism

$$R/_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$$

Examples:

$R = 3\mathbb{Z}$ $\qquad$ $T = \mathbb{Z}_{12}$

① $\quad f : 3\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

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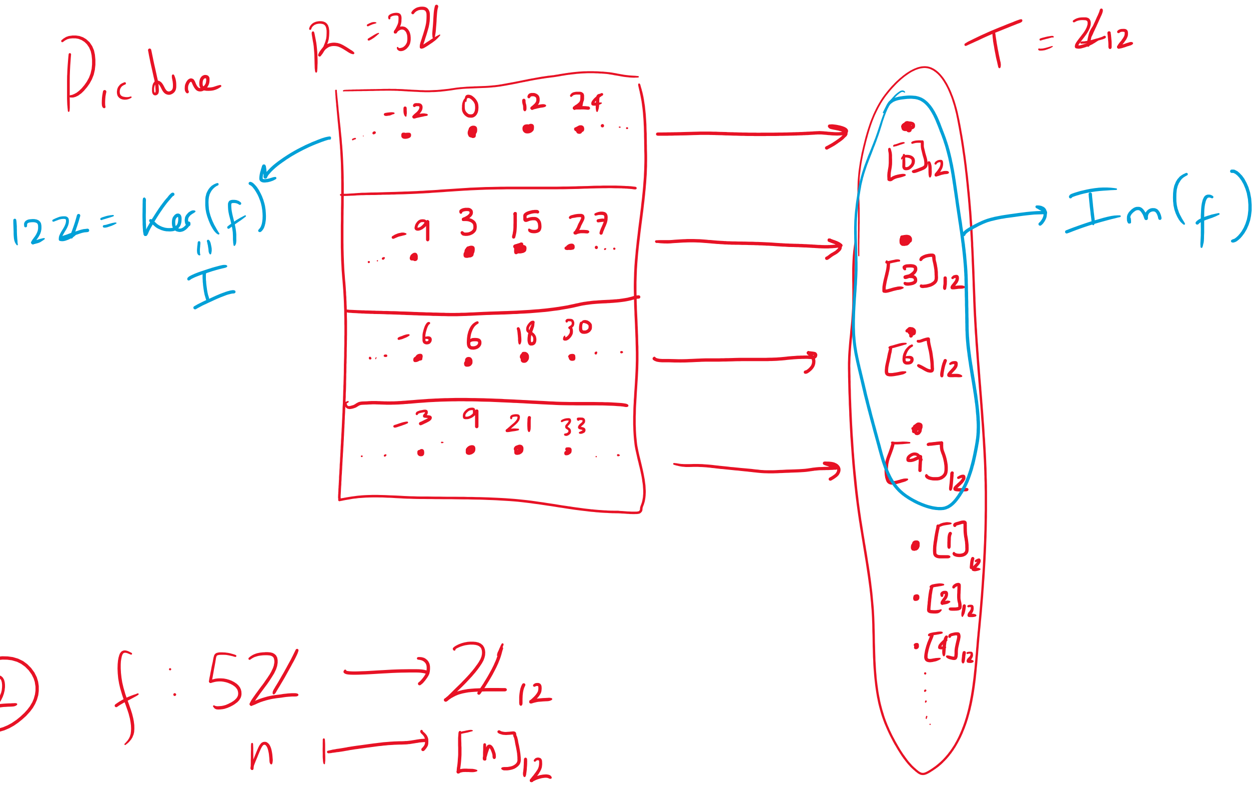
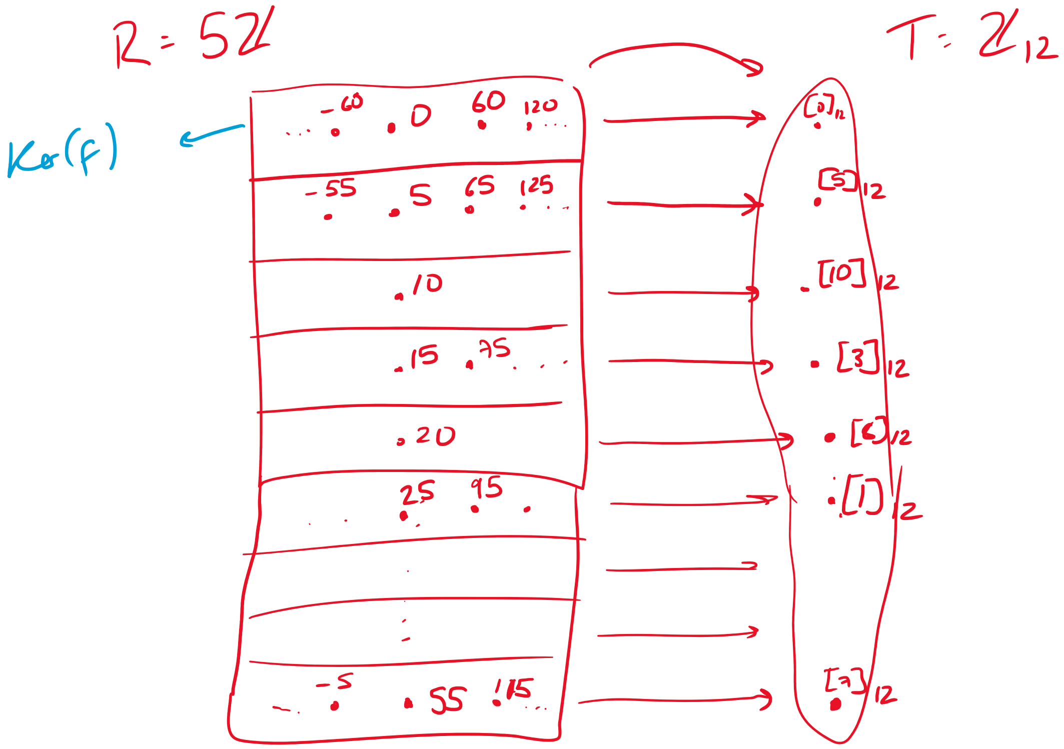
$$f(n) = [n]_{12} .$$

Picture



② $f: 5\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

$n \longmapsto [n]_{12}$



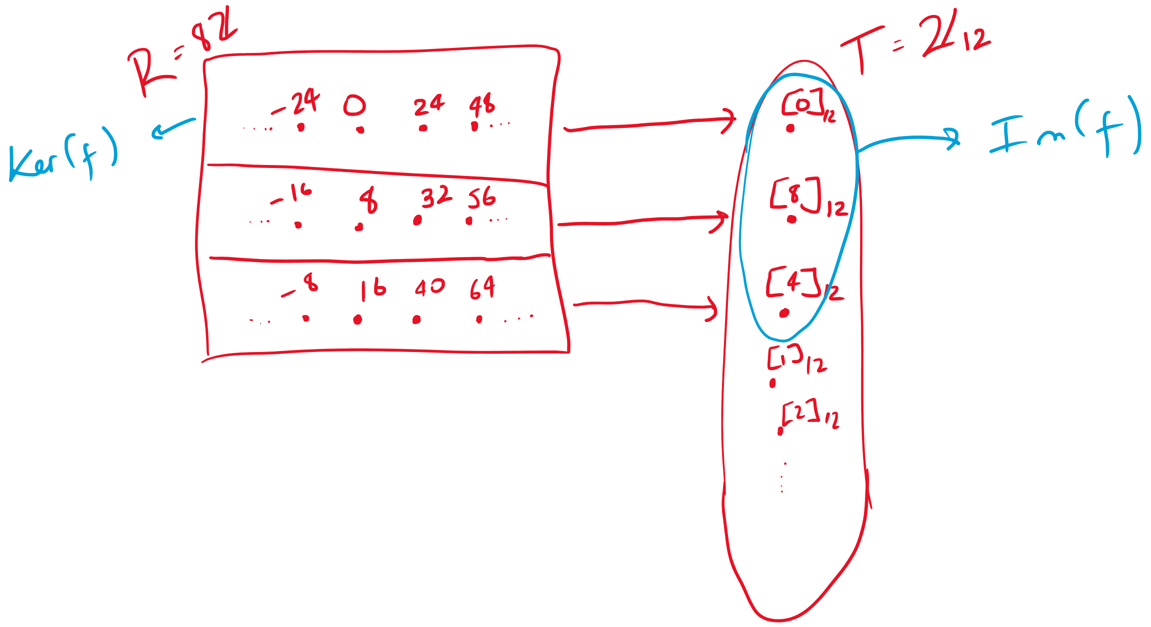
The first isomorphism theorem says

$$5\mathbb{Z}/60\mathbb{Z} \cong \mathbb{Z}_{12}$$

For instance, $5\mathbb{Z}/60\mathbb{Z}$ does have an identity, $[25]_{60\mathbb{Z}}$

(3) $f : 8\mathbb{Z} \longrightarrow \mathbb{Z}_{12}$

$$n \longmapsto [n]_{12}$$



<u>First isomorphism theorem</u>:

$$8\mathbb{Z}/24\mathbb{Z} \cong \{[0]_{12}, [4]_{12}, [8]_{12}\}.$$

$$\left(= 4\mathbb{Z}/12\mathbb{Z}\right)$$

Proof of first isomorphism <u>theorem</u>:

Proof of first isomorphism theorem:

By the first proposition, there is a bijection

$$\varphi : R/_{Ker(f)} \longrightarrow Im(f)$$

sending $\varphi\left([a]_{Ker(f)}\right) \overset{\text{def}}{=} f(a)$

We just need to show that it is a homomorphism.

$$\varphi\left([a]_{Ker} + [b]_{Ker}\right) \overset{\text{addition in } R/I}{=} \varphi\left([a+b]_{Ker}\right)$$

$$\overset{\text{def of } \varphi}{=} f(a+b)$$

$$\overset{\varphi \text{ is homomorphism}}{=} f(a) + f(b)$$

$$\overset{\text{def of } \varphi.}{=} \varphi\left([a]_{Ker}\right) + \varphi\left([b]_{Ker}\right)$$

Same argument to show that $\varphi\left([a]_{Ker} \cdot [b]_{Ker}\right)$

$$= \varphi\left([a]_{Ker}\right) \cdot \varphi\left([b]_{Ker}\right).$$

Example: $R = P(\{a, b, c, d\})$

$T = P(\{a, b\})$

Consider $f : R \longrightarrow T$

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$$f(S) = S \cap \{a,b\}.$$

<u>Claim:</u> This is a homomorphism.

$$f(S_1 + S_2) \overset{?}{=} f(S_1) + f(S_2)$$

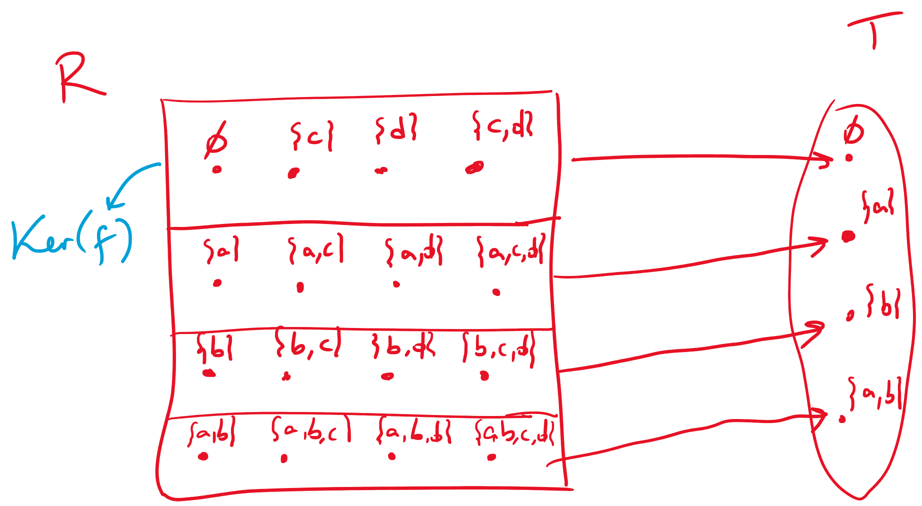
$$(S_1 \triangle S_2) \cap \{a,b\} \overset{?}{=} (S_1 \cap \{a,b\}) \triangle (S_2 \cap \{a,b\}). \quad \checkmark$$

True (by distributivity of $\cap$ over $\triangle$).

$$f(S_1 \cdot S_2) \overset{?}{=} f(S_1) \cdot f(S_2)$$

$$(S_1 \cap S_2) \cap \{a,b\} \overset{?}{=} (S_1 \cap \{a,b\}) \cap (S_2 \cap \{a,b\})$$

True as $a \cdot a = a$.



<u>First isomorphism theorem:</u>

$$\mathcal{P}(\{a,b,c,d\}) / \mathcal{P}(\{c,d\}) \cong \mathcal{P}(\{a,b\})$$