Last time:

We defined Image and Kernel of a homomorphism of rings

We showed that both Image and Kernel are subrings.

In fact, Kernel(f) is an ideal.

- I is an ideal if it is a subring satisfying

  (I2) If $a \in I$ and $b \in R \Rightarrow a \cdot b \in I$.

- Example: Every ring R has two "trivial" ideals.

  $I = \{0\}$ and $I = R$

- Proposition: If R is a ring and I, J are ideals of R then $I \cap J$ is an ideal of R

- Ex. $R = \mathbb{Z}$. Take $I = 4\mathbb{Z}$, $J = 18\mathbb{Z}$.

  We must have $I \cap J$ is an ideal of $\mathbb{Z}$.

$$I \cap J = 36\mathbb{Z}$$

In fact $m\mathbb{Z} \cap n\mathbb{Z} = \operatorname{lcm}(m,n) \cdot \mathbb{Z}$

Proof of proposition:

Proof of proposition:

We want to show that $I \cap J$ is an ideal.

Let's use the ideal test:

(I0) $I \cap J \neq \emptyset$ because $0 \in I \cap J$.

(I1) Suppose $a, b \in I \cap J$. This means $a, b \in I$ and $a, b \in J$. Then $a - b \in I$ and $a - b \in J$, so $a - b \in I \cap J$.

(I2) Suppose $a \in I \cap J$ and $b \in R$. This means $a \in I$ and $a \in J$. Then $a \cdot b \in I$ (because I is an ideal) and also $a \cdot b \in J$ (because J is an ideal), so $a \cdot b \in I \cap J$. ∎

Factor rings (or quotient rings)

Let R be a ring, and I an ideal of R.

We will construct the factor ring R/I (or quotient ring):

- The elements of R/I are the cosets of I in R.
- The operations of R/I are:

$$[a]_I \underset{\substack{\downarrow \\ + \text{ in } R/I}}{+} [b]_I \overset{\text{definition}}{=} [a \underset{\substack{\downarrow \\ + \text{ in } R}}{+} b]_I$$

$+$ in R/I $\qquad$ $+$ in R

$$[a]_I \cdot [b]_I \overset{\text{def}}{=} [a \cdot b]_I$$

$\cdot$ in R/I $\qquad$ $\cdot$ in R.

Ex: $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$

Cosets:

R

A	B	C	D
⋮	⋮	⋮	⋮
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
⋮	⋮	⋮	⋮

In the factor ring:

$$R/I = \{A, B, C, D\}$$

$+$	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

$\cdot$	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	A	C
D	A	D	C	B

Ex: $A = 0$

$-B = D$

$B = 1$

In fact $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$

Thm: The set R/I with that addition and that multiplication is indeed a ring.

- Why are the operations of R/I well-defined? (why do they not depend on the choice of representatives).

Suppose you want to compute

$$[a]_I + [b]_I = [a+b]_I$$

$$[a']_I + [b']_I = [a'+b']_I$$

why equal?

This means

$a - a' \in I$ and $b - b' \in I$

so $a - a' + b - b' \in I$

$$(a+b) - (a'+b') \in I$$

$$\Rightarrow [a+b]_I = [a'+b']_I$$

For multiplication:

Suppose $[a]_I = [a']_I$, $[b]_I = [b']_I$

This means that $a - a' \in I$ and $b - b' \in I$

We want to show that

$$[a \cdot b]_I = [a' \cdot b']_I ,$$

$$[a\cdot b]_I = [a'\cdot b']_I\,,$$

that is, $ab - a'b' \in I$.

To do this, note that $a - a' \in I$

so $(a-a')b \in I$, and $b - b' \in I$

so $a'(b-b') \in I$. (We are using that I is an <u>ideal</u>, not only a subring).

Adding these, we get an element of I:

$$(a-a')b + a'(b-b') = ab - \cancel{a'b} + \cancel{a'b} - a'b'$$
$$= ab - a'b' \in I. \quad ■$$

<u>CW1.</u>

(66) We want to show

$$\mathbb{C} \cong \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

We need to define a bijection

$$f : \mathbb{C} \longrightarrow \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

$$3 + 4i \longmapsto \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

$$-2+i \longmapsto \begin{pmatrix}-2 & 1\\ -1 & -2\end{pmatrix}$$

sum $\quad$ $$1+5i \longmapsto \begin{pmatrix}1 & 5\\ -5 & 1\end{pmatrix}$$

product $\quad$ $$-10-5i \longmapsto \begin{pmatrix}-10 & -5\\ 5 & -10\end{pmatrix}$$

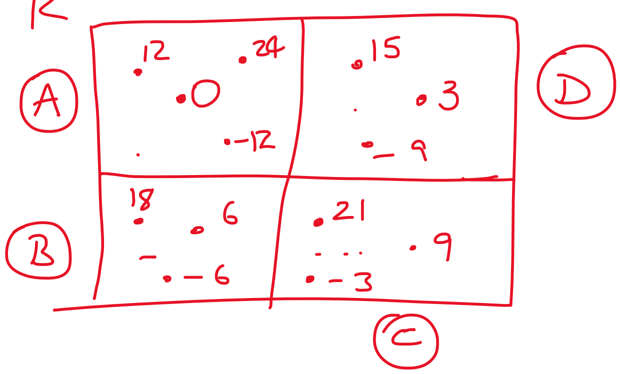
Exercise:

Take $R = 3\mathbb{Z}$ and $I = 12\mathbb{Z}$.

1. Write down the partition of R into cosets of I
2. Explain why I is a ideal of R
3. Write down addition and multiplication tables for R/I.
4. Is R/I isomorphic to a ring we have discussed before?

---

1. $R = 3\mathbb{Z}$ $\qquad$ $I = 12\mathbb{Z}$



Write down the tables.

Find the identity.

Isomorphic to either $\mathbb{Z}_4$ or $P(\{a,b\})$.

Which one?