Last time:

First Isomorphism Theorem:

If $f: R \longrightarrow T$ ring homomorphism

$$R/_{Ker(f)} \cong Im(f)$$

Second Isomorphism Theorem: (understanding the subrings of a quotient ring).

Suppose $R$ is a ring and $I$ an ideal of $R$. There is a one-to-one correspondence between subrings of $R$ that contain $I$ and subrings of the quotient ring $R/I$.

$$\left\{ \begin{array}{l} \text{subrings of } R \\ \text{that contain } I \end{array} \right\} \xleftrightarrow{\text{one-to-one}} \left\{ \text{subrings of } R/I \right\}$$

$$S \text{ subring of } R \text{ containing } I \longleftrightarrow S/I = \{ [s]_I : s \in S \}$$

Moreover, the subring $S$ is an ideal of $R$ if and only if $S/I$ is an ideal of $R/I$.

Example: $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$.

There is a correspondence between

There is a correspondence between

$$\left\{\begin{matrix}\text{subrings of } \mathbb{Z} \\ \text{that contain } 12\mathbb{Z}\end{matrix}\right\} \xleftrightarrow{\text{1-to-1}} \left\{\text{subrings of } \underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{integers mod } 12.}\right\}.$$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
2\mathbb{Z} \arrow[d, dash] \arrow[drr, dash] & & 3\mathbb{Z} \arrow[d, dash] \\
4\mathbb{Z} \arrow[dr, dash] & & 6\mathbb{Z} \arrow[dl, dash] \\
 & 12\mathbb{Z} &
\end{tikzcd}
\quad \xleftrightarrow{\text{1-to-1}} \quad
\begin{tikzcd}[column sep=tiny]
 & \{[0]_{12},[1]_{12},\dots,[11]_{12}\} = \mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z}2/12\mathbb{Z} = \{[0]_{12},[2]_{12},\dots,[10]_{12}\} \arrow[d, dash] \arrow[drr, dash] & & \{[0]_{12},[3]_{12},\dots,[9]_{12}\} = 3\mathbb{Z}/12\mathbb{Z} \arrow[d, dash] \\
4\mathbb{Z}/12\mathbb{Z} = \{[0]_{12},[4]_{12},[8]_{12}\} \arrow[dr, dash] & & \{[0]_{12},[6]_{12}\} = 6\mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \\
 & \{[0]_{12}\} = 12\mathbb{Z}/12\mathbb{Z} &
\end{tikzcd}$$

---

**Corollary:** The subrings of $\mathbb{Z}/m\mathbb{Z}$ correspond precisely to all divisors of $m$:

The divisor $d$ of $m$ corresponds to

the subring $\{[s]_m \mid s \text{ is a multiple of } d\}$

**Example:** How many subrings are there of the ring $\mathbb{Z}/20\mathbb{Z}$?

**Answer:** 6 subrings because 20 has 6 divisors.

$1 \longmapsto \dots$

$2 \longmapsto \dots$

$4 \longmapsto \{[0]_{20},[4]_{20},[8]_{20},[12]_{20},[16]_{20}\}.$

$5 \longmapsto \dots$

$10 \longmapsto \dots$

20 ⟶

For the proof of the 2nd Iso. Thm., look at the textbook.

## Third isomorphism theorem:

Let $R$ be a ring, $S$ a subring of $R$, and $I$ an ideal of $R$.

- $I+S := \{ i+s \mid i \in I \text{ and } s \in S \}$ is a subring of $R$.
- $I \cap S$ is an ideal of $S$.

Thm: There is an isomorphism:

$$S/(I \cap S) \cong (I+S)/I$$

Example: $R = \mathbb{Z}$ $S = 6\mathbb{Z}$ $I = 4\mathbb{Z}$

$$I \cap S = 12\mathbb{Z}$$

$$I+S = \{ i+s \mid i \in 4\mathbb{Z},\ s \in 6\mathbb{Z} \}$$
$$= 2\mathbb{Z}$$

$$S/I\cap S = 6\mathbb{Z}/12\mathbb{Z} = \{ [0]_{12}, [6]_{12} \}$$

$$S/_{I\cap S} = 6\mathbb{Z}/_{12\mathbb{Z}} = \{[0]_{12}, [6]_{12}\}$$

$$^{I+S}/_{I} = 2\mathbb{Z}/_{4\mathbb{Z}} = \{[0]_4, [2]_4\}$$

These are isomorphic rings:

$$6\mathbb{Z}/_{12\mathbb{Z}} \cong 2\mathbb{Z}/_{4\mathbb{Z}} \quad \left(\not\cong \mathbb{Z}/_{2\mathbb{Z}}\right)$$

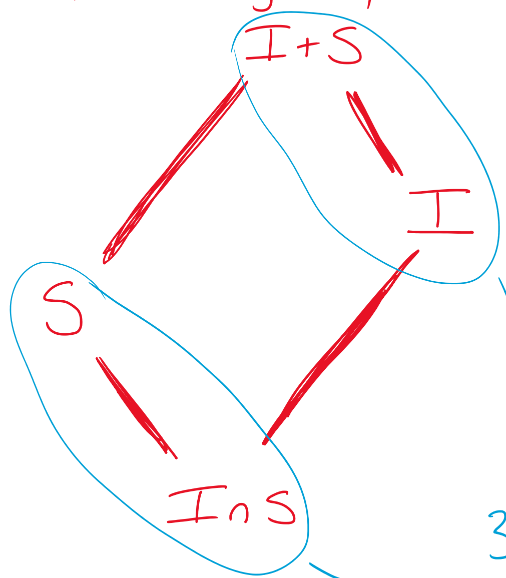
(because here $1 \cdot 1 = 1$)

$$[0]_{12} \longmapsto [0]_4$$
$$[6]_{12} \longmapsto [2]_4$$

---

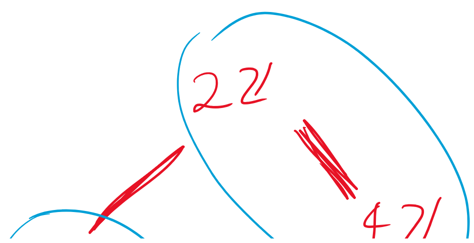
**Picture.** $R$ ring, $S$ subring, $I$ ideal

- Diagram of all subrings of $R$:




3rd Iso. Thm. says these two quotients are isomorphic

Ex:



$$\begin{tikzcd} & & \phantom{x} & \phantom{x} \\ & 6\mathbb{Z} \arrow[ur, dash] \arrow[dr, dash] & & 4\mathbb{Z} \arrow[ul, dash] \arrow[dl, dash] \\ & & 12\mathbb{Z} & \end{tikzcd}$$

isomorphic quotients.

"

In general:

$$\begin{tikzcd} & I+S = \gcd(m,n)\cdot\mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\ S = m\mathbb{Z} \arrow[dr, dash] & & I = n\mathbb{Z} \arrow[dl, dash] \\ & I\cap S = \operatorname{lcm}(m,n)\cdot\mathbb{Z} & \end{tikzcd}$$

isomorphic quotients.

"

For a proof of 3rd Iso. Thm. see textbook.

"

## Coursework 1 Q5.

a) $4^4 = 256$ elements

b) $1 = \begin{pmatrix} \{a,b\} & \emptyset \\ \emptyset & \{a,b\} \end{pmatrix}$

c) Is it commutative?

Answer: No. For instance

$$\begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{a\} \end{pmatrix} \begin{pmatrix} \{a,b\} & \emptyset \\ \emptyset & \{a\} \end{pmatrix} = \begin{pmatrix} \{a\} & \emptyset \\ \{b\} & \{a\} \end{pmatrix}$$

$$\neq \quad \text{not commutative}$$

$$\begin{pmatrix} \{a,b\} & \emptyset \\ \emptyset & \{a\} \end{pmatrix} \cdot \begin{pmatrix} \{a\} & \{b\} \\ \{b\} & \{a\} \end{pmatrix} = \begin{pmatrix} \{a\} & \{b\} \\ \emptyset & \{a\} \end{pmatrix}$$

d) Division ring? No, for instance

$$\begin{pmatrix} \emptyset & \emptyset \\ \emptyset & \{a,b\} \end{pmatrix} \cdot \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} = \begin{pmatrix} \{a,b\} & \emptyset \\ \emptyset & \{a,b\} \end{pmatrix}$$