Lecture Oct 30

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Linear transformations:

Intuition: Model multiplication by a matrix: For instance, think of

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -y \end{pmatrix}.$$

Def: A linear transformation is a function between vector spaces satisfying the following properties:

1. $T(v+w) = T(v) + T(w)$.
2. $T(\alpha v) = \alpha \cdot T(v)$.

In our previous example,

$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $w = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$. Our function is

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ -y \end{pmatrix}.$$

$$v+w = \begin{pmatrix} 6 \\ 10 \end{pmatrix}.$$

$$T(v+w) = \begin{pmatrix} 22 \\ -10 \end{pmatrix}.$$

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \qquad T\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$$

$$T(v) + T(w) = T(v+w). \checkmark$$

Convince yourselves that $T\left(5 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = 5 \cdot T\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Examples:

- Any function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined

- Any function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

  any matrix with m rows, n columns

  is a linear transformation.

<u>Reason</u>: We need to prove the two properties of a linear transformation.

Take two vectors $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

① $$T\Big(\underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{v} + \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{w}\Big) = T\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{pmatrix}\right)$$

$$= A \cdot \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix}$$

$$= A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + A\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$= T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + T\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

② $$T\left(\alpha \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right) = T\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} = A \cdot \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

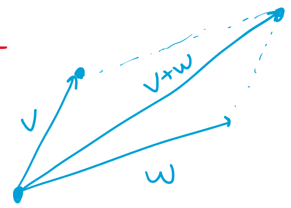
$$= \alpha \cdot A \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \alpha \cdot T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

<u>Ex</u>: The function $T: \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as $T(v) =$ length of v.

is <u>not linear</u> because

is not linear because length of v+w is not equal to the sum of length of v and length of w.



Ex: The function $T: \mathbb{R}^3 \to \mathbb{R}$ defined as

$$T(v) = \underbrace{(3,-1,4)}_{c} \cdot v$$

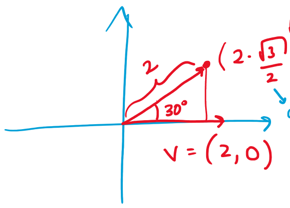
dot product.

is a linear transformation because

① $T(v+w) = c\cdot(v+w) = c\cdot v + c\cdot w$
$= T(v) + T(w)$.

② $T(\alpha v) = c\cdot(\alpha v) = \alpha(c\cdot v) = \alpha T(v)$.

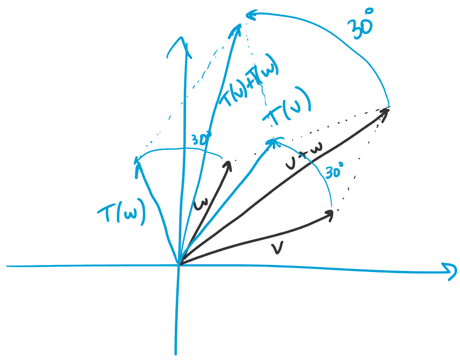
Ex: The function $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(v) =$ rotating v 30° counterclockwise.



$(2\cdot\frac{\sqrt{3}}{2}, 2\cdot\frac{1}{2})$ cos 30° sin 30°

$$T\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}\sqrt{3}\\1\end{pmatrix}.$$

In general, we do have $T(v)+T(w) = T(v+w)$.



and $T(\alpha v) = \alpha T(v)$, so it is a linear transformation.

How do linear transformations "look like"?

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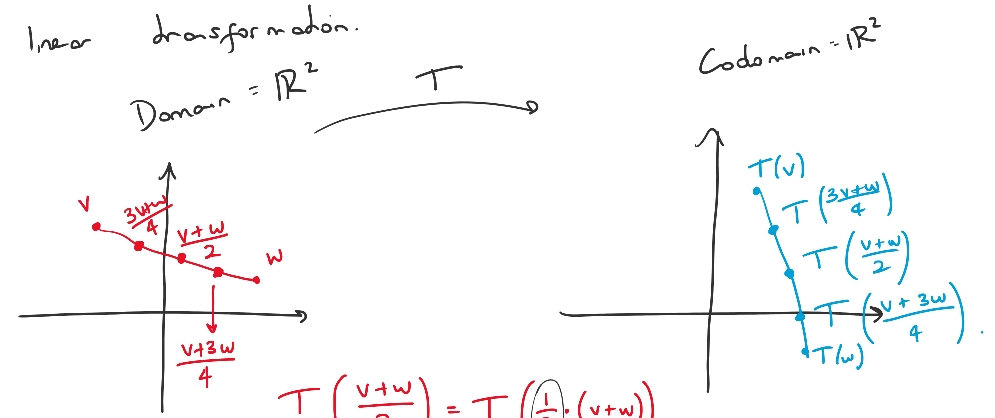
Imagine $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation.



$$T\left(\frac{v+w}{2}\right) = T\left(\frac{1}{2}\cdot(v+w)\right)$$

using property 2 of linear transformations

$$= \frac{1}{2}\, T(v+w)$$

$$= \frac{1}{2}\left(T(v) + T(w)\right).$$

using property 1.

$$= \frac{T(v) + T(w)}{2}.$$

Linear transformations "keep things straight".

Segments are transformed into line segments, triangles are transformed into triangles, and so on.

Exercise: Suppose $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is a linear transformation, satisfying

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

Q: What is $T\begin{pmatrix}1\\1\\0\end{pmatrix}$?

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = T\begin{pmatrix}1\\0\\0\end{pmatrix} + T\begin{pmatrix}0\\1\\0\end{pmatrix}$$

A: $T\begin{pmatrix}1\\1\\0\end{pmatrix} = T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = T\begin{pmatrix}1\\0\\0\end{pmatrix} + T\begin{pmatrix}0\\1\\0\end{pmatrix}$

$= \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$

$= \begin{pmatrix}5\\1\end{pmatrix}.$

Q: What is $T\begin{pmatrix}0\\0\\4\end{pmatrix}$?

A: $T\begin{pmatrix}0\\0\\4\end{pmatrix} = T\left(4\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = 4\cdot T\begin{pmatrix}0\\0\\1\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}.$

Q What is $T\begin{pmatrix}3\\0\\2\end{pmatrix}$?

A: $T\begin{pmatrix}3\\0\\2\end{pmatrix} = T\left(\underline{3\begin{pmatrix}1\\0\\0\end{pmatrix}} + \underline{2\begin{pmatrix}0\\0\\1\end{pmatrix}}\right)$

$= T\left(3\begin{pmatrix}1\\0\\0\end{pmatrix}\right) + T\left(2\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$

$= 3\cdot T\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\,T\begin{pmatrix}0\\0\\1\end{pmatrix}$

$= 3\cdot\begin{pmatrix}2\\-1\end{pmatrix} + 2\cdot\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}6\\-3\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}8\\1\end{pmatrix}.$

Fact: If T is a linear transformation then

$$T(\underbrace{a_1}_{}v_1 + \underbrace{a_2}_{}v_2 + \dots + \underbrace{a_n}_{}v_n) = a_1\cdot T(v_1) + a_2 T(v_2) + \dots + a_n T(v_n)$$

scalars.

Q: What is $T\begin{pmatrix}1\\2\\3\end{pmatrix}$?

Q: What is $T\begin{pmatrix}1\\2\\3\end{pmatrix}$?

$$T\begin{pmatrix}1\\2\\3\end{pmatrix} = T\left(1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= 1\cdot T\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\,T\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\,T\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= 1\cdot\begin{pmatrix}2\\-1\end{pmatrix} + 2\cdot\begin{pmatrix}3\\2\end{pmatrix} + 3\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}6\\4\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}11\\9\end{pmatrix}.$$

In general, what is

$$T\begin{pmatrix}a\\b\\c\end{pmatrix} = T\left(a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= a\cdot T\begin{pmatrix}1\\0\\0\end{pmatrix} + b\,T\begin{pmatrix}0\\1\\0\end{pmatrix} + c\,T\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= a\cdot\begin{pmatrix}2\\-1\end{pmatrix} + b\cdot\begin{pmatrix}3\\2\end{pmatrix} + c\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

$$= \begin{pmatrix}2a+3b+c\\-a+2b+2c\end{pmatrix} = \begin{pmatrix}2 & 3 & 1\\-1 & 2 & 2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

Theorem: Any linear transformation

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

can be written as

$$T(v) = A \cdot v.$$

The matrix A is obtained by putting the values of $T(e_1)$, $T(e_2)$, $\dots$, $T(e_n)$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$A = {}_m\overset{n}{\begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}}$$

<u>Ex:</u> Consider the linear transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

defined as

$T(v) =$ rotating the vector v θ radians counterclockwise.

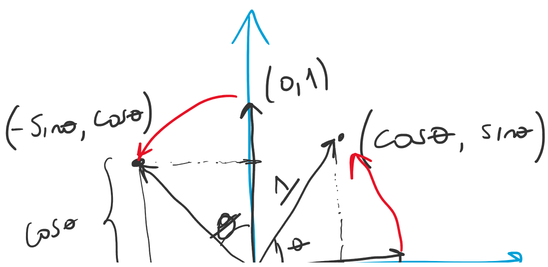
Our theorem says that we can also write

$$T(v) = \begin{pmatrix} & A & \end{pmatrix} \cdot v.$$

where

$$A = \begin{pmatrix} | & | \\ T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \end{pmatrix}$$



$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$