Lecture Oct 26

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Span:

- If v is a vector, $\text{span}(v)$ = line in direction v.

- If v, w are vectors,

$\text{span}(v, w) =$
  - usually the plane containing v and w if v, w are independent
  - if one is a multiple of the other then it is the line containing both v, w.

$\text{span}(v, w, u) =$
  - usually it is the 3D space where they live if they are independent
  - If they live all in a plane (but are not collinear) their span is the plane containing them
  - If they all live in a line the span is that line

Rank and nullity of a matrix.

Suppose

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 3 & 5 \end{pmatrix}$$

$m \times n$ matrix

Spaces you can produce from this matrix:

- <u>Null space</u>: Solutions to $A\vec{x} = 0$.
  Subspace in $\mathbb{R}^{n}$. $n \rightarrow$ # of columns.
- <u>Column space</u>: Subspace spanned by the columns. It is a subspace of $\mathbb{R}^{m}$ $m \rightarrow$ # of rows
- <u>Row space</u>: Subspace spanned by the rows. It is a subspace of $\mathbb{R}^{n}$ $n \rightarrow$ # of columns.

<u>How do we find bases for these spaces?</u>

- For the nullspace, solve the system

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 1 & 2 & 4 & 6 & 0 \\ 1 & 2 & 3 & 5 & 0 \end{array}\right)$$

$\downarrow$ RREF

pivots

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \end{array}\right.$$

pivots

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

free variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

basis for the nullspace.

$\boxed{\text{Dimension of the nullspace is the number of free variables.}}$

This number is called the <u>nullity</u> of the matrix. In our example

$$\text{nul}(A) = 2.$$

- For the column space, take the original columns corresponding to pivot variables. In our case, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \end{pmatrix} \begin{pmatrix} 3 \end{pmatrix} \right.$$

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \right\}.$$

Dimension of the column space is the number of pivots

This number is called the rank of the matrix. In our example

$$\text{rank}(A) = 2.$$

Rank-nullity theorem

- For any matrix

$$\text{rank}(A) + \text{nul}(A) = \#\text{ of columns of } A$$

- To find a basis for the rowspace of A, take the nonzero rows in the (reduced) echelon form. In our example

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$ is a basis for the rowspace.

$$\boxed{\text{Dimension of the rowspace is also the number of pivots.}}$$