Lecture Oct 23

23 October 2023 10:00

Last time:

- **Span**: Space generated by some collection of vectors.
- **Linear independence**: A collection of vectors is independent if "none of the vectors are redundant".
- **Basis**: A collection of vectors is a basis for a vector space V if they span V and are linearly independent. (They span it without redundance).

Dimension

Ex: Find a basis for the plane in $\mathbb{R}^3$ defined by the equation $3x - 4y + z = 0$.

$$\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

dot product

Note that this is equivalent to solving the system $\begin{pmatrix} 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

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$$\begin{pmatrix} 3 & -4 & 1 \end{pmatrix}.$$

We can find a basis by reducing to row-echelon form.

In this way we find a basis consisting of 2 vectors.

A basis for this plane consists of two vectors in the plane that are not one a multiple of the other. There are lots of bases!

Fact: Any two bases for the same vector space have the same number of elements. This number is called the dimension of the vector space.

- What is the dimension of $\mathbb{R}^n$?

One basis for $\mathbb{R}^n$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = x_1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_1} + x_2 \cdot \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_2} + x_3 \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_3} + \dots + x_n \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{e_n}$$

This means that $e_1, e_2, \dots, e_n$ span the vector space $\mathbb{R}^n$, And they are linearly independent, so they are a basis for $\mathbb{R}^n$.

This is sometimes called the <u>standard basis</u> for $\mathbb{R}^n$.

$\boxed{\text{The dimension of } \mathbb{R}^n \text{ is } n.}$

What are the bases of $\mathbb{R}^n$?

<u>Ex</u>: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$ are they a basis for $\mathbb{R}^3$?

They are linearly independent, but they do not span all of $\mathbb{R}^3$. They only do not span all of $\mathbb{R}^3$. They only span a plane in $\mathbb{R}^3$.

(The plane has normal vector $\begin{pmatrix}1\\2\\4\end{pmatrix} \times \begin{pmatrix}7\\-1\\3\end{pmatrix}$)

They are not a basis for $\mathbb{R}^3$, but they are a basis for that plane.

$$\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix} = \hat{i}(6+4) - \hat{j}(3-28) + k(-1-14)$$

$$= 10\hat{i} + 25\hat{j} - 15\hat{k}.$$

Cross-product of the two vectors

Any basis of $\mathbb{R}^3$ needs to have exactly 3 vectors.

Sweet spot

Intuition:

1 vector   2 vectors   3 vectors ...   dim many vectors   dim+1 vectors   dim+2 vectors

not enough to span the vector space

linearly dependent.

• <u>Question</u>: When is a collection of vectors $v_1, v_2, \dots, v_k$ a basis of $\mathbb{R}^n$?

• <u>Answer</u>: There need to be exactly n many vectors, And the matrix

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has all columns with pivots, meaning the matrix is invertible.

$$\boxed{\text{Bases of } \mathbb{R}^n \overset{\text{same}}{\leftrightsquigarrow} \text{invertible } n\times n \text{ matrices}}$$

<u>Why are bases important?</u>

• If V is a vector space and $v_1, \dots, v_n$ form a basis for V, then any element $v \in V$ can be written <u>uniquely</u> in the form

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n$$

Scalars.

called the coordinates of v in this basis.

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(If you could write v in two different ways as a combination of $v_1, \dots, v_n$, that would be giving a dependence among the $v_1, \dots, v_n$).

Ex: $v = 3v_1 + 7v_2 + 2v_3$

$= -2v_1 + 0v_2 - 1v_3$

not possible because $v_1, \dots, v_n$ are independent.

Ex: $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 10x + 25y - 15z = 0 \right\}$.

A basis for V is $\left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}_{v}, \underbrace{\begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}}_{w} \right\}$

$$\begin{pmatrix} 15 \\ 0 \\ 10 \end{pmatrix} = 1\,v + 2\,w$$

coordinates of $\begin{pmatrix} 15 \\ 0 \\ 10 \end{pmatrix}$ in the basis $\{v, w\}$.

Spaces of functions and matrices.

• Ex: The space of polynomials of degree at most 2 is a vector space. Ex: x^2-5x+1, $x+7$, x^2-2, $-x^2+3x$

(Q: Is the space of polynomials of degree equal to 2 a vector space?)
No: $(x^2+5)+(-x^2+3x)$ is not degree $=2$

A basis for this vector space is:

$$x^2, \quad x, \quad 1.$$ ← canonical basis

Another possible basis:

$$x^2, \quad x^2+1, \quad x^2+x$$

The dimension of the space of polynomials of degree at most 2 is 3.

Ex: Solutions to a homogeneous linear differential equation of degree 2.

are a vector space. of dimension 2.

Ex: $y'' + 7y' - y = 0$.

Ex: $y'' - y = 0$. $\boxed{y'' = y}$

Solutions: e^x, e^{-x} ← basis.

General solution $f(x) = Ae^x + Be^{-x}$.