Span

Definition: The vectors $v_1, v_2, \dots, v_k$ span or generate the vector space V if every element of V can be generated with the vectors $v_1, \dots, v_k$

$$\text{span}(v_1, \dots, v_k) = \{ w \in V \mid w = x_1 v_1 + x_2 v_2 + \dots + x_k v_k \}$$

scalars

Ex: Do the vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ span all of $\mathbb{R}^3$?

Answer: No, because $\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = xy$ plane

$$= \{ w \in \mathbb{R}^3 \mid w_3 = 0 \}.$$

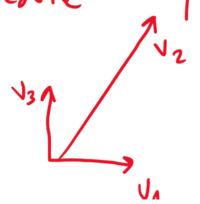
Linear independence:

Def: The vectors $v_1, \dots, v_k$ are called linearly independent if "there are no linear dependences among them", that is

$x_1 v_1 + x_2 v_2 + \dots + x_k v_k = 0$ only when all the x_is are 0.

Ex: Are the vectors $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}2\\3\\0\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_3}$ linearly independent or dependent?

Picture xy plane



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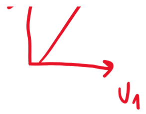

$$2v_1 - v_2 + 3v_3 = 0 \longleftarrow \boxed{v_2 = 2v_1 + 3v_3}$$

so v_1, v_2, v_3 are linearly dependent.

If we remove v_2 we get

v_1, v_3 which still span the same space

but v_1, v_3 are now linearly independent.

Fact: $$v \in \operatorname{span}(v_1, v_2, \dots, v_k)$$
$$\Updownarrow$$
$$\operatorname{span}(v_1, \dots, v_k, v) = \operatorname{span}(v_1, \dots, v_k).$$

Test for linearly independence

Ex: Are the vectors
$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 0 \\ 6 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -3 \\ -1 \end{pmatrix}$$
linearly dependent or independent?

We need to find the solutions to
$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

To solve this,
$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & 0 & -3 & 0 \\ 7 & 6 & -1 & 0 \end{array}\right)$$

Test for independence

linearly independent if all variables are

Test for independence

linearly independent if all variables are pivot variables.

linearly dependent if there are some free variables.

Basis

Def: Some collection of vectors $v_1, \dots, v_k$ form a basis for a vector space V if

- $v_1, \dots, v_k$ span all of V
- $v_1, \dots, v_k$ are linearly independent.

Ex: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span the xy-plane but they are dependent so they are not a basis for the xy-plane.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span the xy-plane and now are linearly independent so they are a basis for the xy-plane

Ex: Compute a basis for the column space of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 3 & 5 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 1 & 2 & 3 & 5 \end{pmatrix}.$$

<u>Ans</u>: The column space is generated by the columns

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}.$$
$$\quad v_1 \qquad\quad v_2 \qquad\quad v_3 \qquad\quad v_4$$

They are dependent: $2v_1 = v_2$

$$\boxed{2v_1 - v_2 = 0.}$$

To get rid of all dependences, we put the vectors in a matrix and we reduce the matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 3 & 5 \end{pmatrix}$$
$$\downarrow$$
$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\downarrow$$
redundant.

$$\begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leftarrow \text{row reduced echelon form}$$

redundant

The columns ~~have~~ changed, but <u>the dependences among the columns have not changed!</u>

A basis for the column space of the original matrix is obtained by just taking the original columns that correspond to pivots.

In this case $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$

are a basis for the column space.