Lecture Nov 30

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Orthogonal complements:

Definition: If L is a subspace of $\mathbb{R}^n$, the orthogonal complement $L^\perp$ is the set of vectors that are orthogonal to all vectors in L.

$$L^\perp = \{ w \in \mathbb{R}^n \mid w \cdot v = 0 \text{ for all } v \in L \}$$

Properties:

- $L^\perp$ is a subspace of $\mathbb{R}^n$.
- A vector w is in $L^\perp$ if and only if it is orthogonal to the set of vectors in any spanning set of L.

Ex: Suppose L is the subspace

$$L = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right\}.$$

Compute $L^\perp$.

Solution: A vector is in $L^\perp$ if and only if it is orthogonal to $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

only if it is orthogonal to $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-2\\3\end{pmatrix}$. This means that a vector $\begin{pmatrix}x\\y\\z\end{pmatrix}$ is in $L^\perp$ if

$$0 = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = 1 \cdot x - 1y + 2z$$

$$0 = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}2\\-2\\3\end{pmatrix} = 2x - 2y + 3z.$$

We solve this system

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & -2 & 3 & 0 \end{array}\right)$$

$$\Big\downarrow \text{reduce to RREF}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{pivot} & \text{free variable} & \text{pivot} \end{array}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}y\\y\\0\end{pmatrix} = y\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$L^\perp = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}. \longleftarrow \text{line in } \mathbb{R}^3.$$

Note: For any matrix A, we have

Note: For any matrix A, we have

- $\boxed{\operatorname{col}(A)^{\perp} = \operatorname{Nullspace}(A^T)}.$
- $\operatorname{rowspace}(A)^{\perp} = \operatorname{Nullspace}(A).$

Orthogonal sets.

Definition: A collection $\{v_1, v_2, \dots, v_k\}$ of vectors in $\mathbb{R}^n$ is called an <u>orthogonal set</u> if any two distinct vectors v_i, v_j are orthogonal, that is,

$$v_i \cdot v_j = 0 \quad \text{for all } i \neq j.$$

Example:

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} \right\}$$

is an orthogonal set.

Proposition:

If $\{v_1, \dots, v_k\}$ is an orthogonal set of nonzero vectors then $v_1, \dots, v_k$ are linearly independent.

Proof: Suppose the vectors $v_1, \dots, v_k$

Reason: Suppose the vectors $v_1, \dots, v_k$ satisfy

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k v_k = \vec{0}$$

Taking dot product with v_1 on both sides:

$$v_1 \bullet (a_1 v_1 + a_2 v_2 + \dots + a_k v_k) = v_1 \bullet \vec{0} = \underset{\text{Scalar}}{0}$$

Using the properties of dot product,

$$a_1 \, v_1 \bullet v_1 + a_2 \cancelto{0}{v_1 \bullet v_2} + a_3 \cancelto{0}{v_1 \bullet v_3} + \cancelto{0}{\dots} + a_k \cancelto{0}{v_1 \bullet v_k} = 0$$

because they are orthogonal

$$a_1 \underbrace{v_1 \bullet v_1}_{\text{length of } v_1 \neq 0.} = 0$$

$$a_1 = 0 .$$

By repeating this argument with all other vectors $v_2, v_3, \dots v_k$. we conclude that

$$a_2 = 0$$
$$a_3 = 0$$
$$\vdots$$
$$a_k = 0$$

This means that the vectors $v_1, \dots, v_k$

are linearly independent. $\blacksquare$

<u>Definition</u>: An <u>orthogonal basis</u> for a subspace L is a basis for L that is also an orthogonal set.

<u>Ex</u>: Can we find an <u>orthogonal</u> basis for the subspace of $\mathbb{R}^3$ described by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = x + 2y - z = 0 \quad ?$$

We want to find two vectors in that plane that are orthogonal to each other.

For instance

$$\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad ?$$

some other vector in the plane but perpendicular to the first one.