Lecture Nov 27

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Recall:

Suppose $A \in \mathbb{R}^{n\times n}$ and D is an $n\times n$ diagonal matrix, P is an $n\times n$ invertible matrix such that

$$A = P \cdot D \cdot P^{-1}$$

In this case, A is called a diagonalisable matrix, and

$$P = \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}}_{\text{eigenvectors of } A} \qquad D = \underbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}_{\text{corresponding eigenvalues.}}$$

Example:

Can you diagonalise

$$A = \begin{pmatrix} -6 & 3 & -2 \\ -7 & 5 & -1 \\ 8 & -3 & 4 \end{pmatrix}$$

Sol: We compute the eigenvalues of A

$$\det(A - \lambda I) = \det \begin{pmatrix} -6-\lambda & 3 & -2 \\ -7 & 5-\lambda & -1 \end{pmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} -7 & 5-\lambda & -1 \\ 8 & -3 & 4-\lambda \end{pmatrix}$$

$$= -(\lambda+1)(\lambda-2)^2.$$

The eigenvalues are $\lambda=-1$ and $\lambda=2$.

The eigenvectors of A are computed as

$$\text{Nullspace}\,(A-(-1)I) = \text{span}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}. \leftarrow \dim = 1$$

$$\text{Nullspace}\,(A-2I) = \text{span}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}. \leftarrow \dim = 1$$

instead of $\dim = 2$

There are not enough linearly independent eigenvectors to construct the invertible matrix P. The matrix A is not diagonalisable

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- In general, if A is an $n\times n$ matrix:

Characteristic equation

Characteristic equation

$$\underbrace{\det(A - \lambda I)}_{\text{polynomial of degree } n.} = 0$$

You might need complex numbers to find all solutions.

$$\boxed{\begin{gathered}\underline{x^2+1} = 0 \\ (x+i)(x-i) = 0\end{gathered}}$$

If you factor it looks like

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_K)^{m_K} = 0$$

When computing the corresponding eigenspaces

Nullspace $(A - \lambda_1 I)$ $\leftarrow$ dimension between 1 and m_1

Nullspace $(A - \lambda_2 I)$ $\leftarrow$ dimension between 1 and m_2

$\vdots$

Nullspace $(A - \lambda_k I)$ $\leftarrow$ dimension between 1 and m_k.

The matrix A is diagonalisable precisely when you find the maximum number of eigenvectors in every case.

Application: A formula for the Fibonacci sequence.

Def: The Fibonacci sequence is defined as

$F_0 = 0 \quad F_1 = 1 \qquad F_{n+1} = F_n + F_{n-1}$

$F_2 = 1 \quad F_3 = 2 \quad F_4 = 3 \quad F_5 = 5 \quad F_6 = 8$

$F_7 = 13 \quad F_8 = 21 \quad F_9 = 34 \quad F_{10} = 55 \quad F_{11} = 89 \quad \ldots$

Note that:

$$\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}$$

This means

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\vdots$$

In general

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

To understand $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$, let's diagonalise it.

Characteristic equation : $\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0.$$

Using the quadratic formula:

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{golden ratio}$$

$$\varphi' = \frac{1-\sqrt{5}}{2}$$

The caracteristic equation:

$$\left(\lambda - \frac{1+\sqrt{5}}{2}\right)\left(\lambda - \frac{1-\sqrt{5}}{2}\right) = 0.$$

All exponents are 1 so the matrix is diagonalisable. Computing the eigenvectors we get

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \varphi & \varphi' \\ 1 & 1 \end{bmatrix}}_{P} \cdot \underbrace{\begin{bmatrix} \varphi & 0 \\ 0 & \varphi' \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} 1 & -\varphi' \\ -1 & \varphi \end{bmatrix} \cdot \frac{1}{\det(P)}}_{P^{-1}} .$$

$$\det P = \varphi - \varphi' = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}.$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \cdot \begin{bmatrix} \varphi & \varphi' \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \varphi' \end{bmatrix} \begin{bmatrix} 1 & -\varphi' \\ -1 & \varphi \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = A^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left(P D P^{-1}\right)^n \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= P \cdot D^n \cdot P^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \varphi' \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \varphi'^n \end{bmatrix} \begin{bmatrix} 1 & -\varphi' \\ -1 & \varphi \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \varphi' \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \varphi'^n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \varphi' \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n \\ -\varphi'^n \end{bmatrix}$$

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - (\varphi')^{n+1} \\ \varphi^n - (\varphi')^n \end{bmatrix}$$

We conclude

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - (\varphi')^n \right) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Orthogonality:

Def: If $v, w \in \mathbb{R}^n$, we say that they are **orthogonal** if $\left(v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \right)$

def dot of product

$$v \cdot w = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = 0$$

def of orthogonal.

not matrix multiplication

Note that:

$$v \cdot w = \begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = v^T w.$$

$$V \bullet W = \begin{pmatrix} V_1 & V_2 & \dots & V_n \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{pmatrix} = \underset{\text{matrix multiplication}}{V' W}.$$

Properties:

ⓐ $V \bullet W = W \bullet V$

ⓑ $(U + V) \bullet W = U \bullet W + V \bullet W$

ⓒ $(\alpha V) \bullet W = \alpha (V \bullet W)$

α scalar

ⓓ $V \bullet V = V_1^2 + V_2^2 + \dots + V_n^2 \geq 0$

And $V \bullet V = 0 \iff V = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Def: The length (or the norm) of a vector V is

$$\|V\| = \sqrt{V_1^2 + V_2^2 + \dots + V_n^2}$$

$$= \sqrt{V \bullet V}.$$

$\|\alpha V\|$?

Exercise. What is $\|\alpha v\|$?

$$\|\alpha v\| = \sqrt{\alpha^2 v_1^2 + \alpha^2 v_2^2 + \dots + \alpha^2 v_n^2} = \sqrt{\alpha^2 (v_1^2 + v_2^2 + \dots + v_n^2)}$$

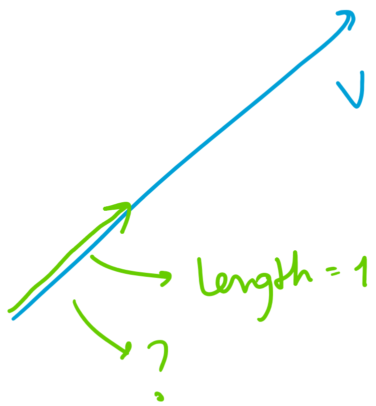
$$= |\alpha| \cdot \sqrt{v_1^2 + \dots + v_n^2}$$

$$= |\alpha| \cdot \|v\| .$$

Exercise: What is the vector in the same direction as v but with length equal to 1?

Answer.

$$\frac{v}{\|v\|} \leftarrow \text{scalar.}$$

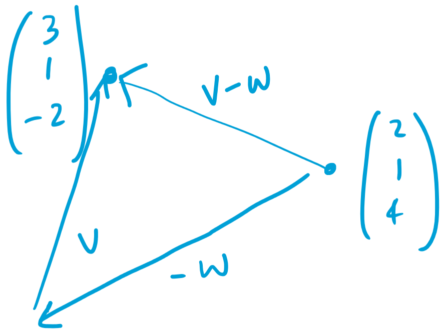


This is called normalising the vector v.

Definition The distance between vectors v and w is defined as

$$d(v, w) = \|v - w\|$$

$$\text{dist}(v,w) = \|v-w\|$$



Exercise: (Pythagorean theorem)

v and w are orthogonal vectors

$$\iff \|v+w\|^2 = \|v\|^2 + \|w\|^2 .$$

Solution.

$$\|v+w\|^2 = \sqrt{(v+w)\cdot(v+w)}^{\,2} = (v+w)\cdot(v+w)$$

$$= v\cdot v + \underline{v\cdot w} + \underline{w\cdot v} + w\cdot w$$

$$= \|v\|^2 + 2(v\cdot w) + \|w\|^2$$

$$= \|v\|^2 + \|w\|^2 \quad \text{precisely when } v\cdot w = 0$$

(v, w are

$= \|v\| + \|w\|$

(meaning v, w are orthogonal).