Lecture Nov 23

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Recap:

If A is an $n \times n$ matrix and $v \in \mathbb{R}^n$ satisfy

$$A \cdot v = \lambda \cdot v$$

then λ is an eigenvalue and v is an eigenvector.

We can compute eigenvalues by solving the characteristic equation

$$\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial of } A\text{, it has degree } n} = 0 .$$

Given a particular eigenvalue λ, its set of eigenvectors is called an eigenspace, equal to

$$\text{Nullspace}(A - \lambda I).$$

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Note: Computing powers of a matrix

Note: Computing powers of a matrix is hard in general. However, if D is a diagonal matrix then

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad D^{10} = \begin{pmatrix} 3^{10} & 0 & 0 \\ 0 & 7^{10} & 0 \\ 0 & 0 & (-2)^{10} \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}$$

Last time we computed the eigenvalues

$$\lambda = -1 \qquad \lambda = 2$$

For $\lambda = -1$, the corresponding eigenspace is

$$\operatorname{span}\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{v_1}.$$

For $\lambda = 2$, the corresponding eigenspace is

$$\operatorname{span}\underbrace{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}_{v_2}$$

Now, consider $P = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}$

Now, consider $P = \begin{array}{cc} \begin{pmatrix} 1 & 3 \end{pmatrix} \\ \downarrow \quad \downarrow \end{array}$

eigenvectors v_1, v_2.

$$A \cdot P = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix} \cdot \begin{array}{c} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ v_1 & v_2 \end{array} \end{array}$$

$$= \begin{pmatrix} | & | \\ Av_1 & Av_2 \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} | & | \\ -1 \cdot v_1 & 2 v_2 \\ | & | \end{pmatrix}$$

Take $D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ ← diagonal matrix with eigenvalues as entries.

Let's compute

$$P \cdot D = \begin{array}{c} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \\ \begin{array}{cc} \downarrow & \downarrow \\ v_1 & v_2 \end{array} \end{array} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$= \begin{pmatrix} | & | \\ -1 \cdot v_1 & 2 \cdot v_2 \\ | & | \end{pmatrix}$$

<u>Theorem</u>: If A is an $n \times n$ matrix, and

$$D = \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}}_{\text{matrix of eigenvalues}} \qquad P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

$\downarrow \downarrow \downarrow$ corresponding eigenvectors

then $$A \cdot P = P \cdot D.$$

If P is an invertible matrix, (that is, if the eigenvectors $v_1, \dots, v_n$ are linearly independent), then

$$A = P \cdot D \cdot P^{-1}.$$

If this is the case, A is called a <u>diagonalisable</u> matrix.

Ex: In our previous example

$$A = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}$$

The eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ are linearly independent so P is an invertible matrix.

We have

$$A = P D P^{-1}$$

$$\begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}^{-1}$$

This could be useful to compute

$$A^{10} = \underbrace{(P D \cancel{P^{-1}})}_{A} \underbrace{(\cancel{P} \cdot D \cdot \cancel{P^{-1}})}_{A} \underbrace{(\cancel{P} \cdot D \cancel{P^{-1}})}_{A} \cancel{\;} \cdots \cancel{\;} \underbrace{(\cancel{P} D P^{-1})}_{A}$$

$$= P D^{10} P^{-1}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} (-1)^{10} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}^{-1}.$$

$$= \underbrace{\begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} (-1)^{[illegible]} & 0 \\ 0 & 2^{10} \end{pmatrix} \begin{pmatrix} [illegible] & [illegible] \\ 1 & 3 \end{pmatrix}}_{\text{easy to compute.}} .$$

Ex. Diagonalise (or explain why it is not possible) the following matrix

$$A = \begin{pmatrix} -7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5 \end{pmatrix}.$$

We compute eigenvalues and eigenvectors.

Charteristic equation:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -7-\lambda & 3 & -3 \\ -9 & 5-\lambda & -3 \\ 9 & -3 & 5-\lambda \end{pmatrix}$$

$$\overset{\text{computation}}{=} -\lambda^3 + 3\lambda^2 - 4 \overset{\text{long division}}{=} -(\lambda+1)(\lambda-2)^2$$

Two eigenvalues $\lambda = -1$ $\lambda = 2$.

The corresponding eigenspaces are:

For $\lambda = -1$, Nullspace $(A - (-1)I) =$

$$\text{Span}\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$, Nullspace $(A - 2I)$

$$= \text{span}\left(\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right).$$

To diagonalise A, we construct

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

We get that P is an invertible matrix

and $$A = P \cdot D \cdot P^{-1}$$