Lecture Nov 20

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Eigenvalues and eigenvectors

Example: Suppose $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

For some special vectors v, the result $A \cdot v$ is very simple

If $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $Av = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

$= 3v$.

If $w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ then $Av = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$= 2w$.

Definition: Suppose A is an $n \times n$ matrix and $v \in \mathbb{R}^n$ is a non-zero vector such that

vector with n coordinates

$$Av = \lambda v$$

$n \times n$ matrix

Scalar

In this case, λ is called an eigenvalue of A and v is an associated eigenvector.

Ex: Take $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$.

Is $v = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ an eigenvector of A?

Let's compute $Av = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -24 \\ 20 \end{pmatrix}$

$= -4v$.

- v is an eigenvector of A and -4 is its corresponding eigenvalue.

Q Is $w = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ an eigenvector of A?

No, $A \cdot w$ is not a multiple of w.

Example: $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$. Show that $\exists$ is an eigenvalue of this matrix.

Solution: We want to find a vector $v \in \mathbb{R}^2$

Such that $\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} v = 7v.$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} v - 7v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} v - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-7\cdot 1 & 6 \\ 5 & 2-7\cdot 1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To find such a v, we need to find the nullspace of $\begin{pmatrix} 1-7\cdot 1 & 6 \\ 5 & 2-7\cdot 1 \end{pmatrix}$.

$$\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \xrightarrow[\underbrace{R_2 = R_2 - 5R_1}]{R_1 = R_1/-6} \begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$$

$\uparrow$ pivot variable $\qquad$ $\uparrow$ free variable

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Nullspace $A = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

Any vector in the nullspace will be an eigenvector of the matrix.

For instance

$$\begin{pmatrix}1 & 6\\5 & 2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}7\\7\end{pmatrix} = 7\begin{pmatrix}1\\1\end{pmatrix}.$$

Theorem: Suppose $A \in \underbrace{\mathbb{R}^{n\times n}}_{\text{space of } n\times n \text{ matrices with real entries.}}$ and λ is a scalar.

Then the following statements are all equivalent:

- λ is an eigenvalue of A.
- $(A-\lambda\cdot I)v = 0$ has a non-trivial solution
- Nullspace $(A-\lambda I)$ is more than just $\{0\}$.
- $A-\lambda I$ has at least one free variable when reduced to row-echelon form.

- $A - \lambda I$ is not invertible
- $\det(A - \lambda I) = 0$.

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Example: $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$. Show that $\lambda = 7$ is an eigenvalue of A.

Solution: Compute

$$\det(A - 7I) = \det\begin{pmatrix} 1-7 & 6 \\ 5 & 2-7 \end{pmatrix}$$

$$= \det\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} = 30 - 30 = 0.$$

We conclude that 7 is an eigenvalue.

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Definition: If $A \in \mathbb{R}^{n \times n}$ then the equation

$$\det(A - \underset{\text{variable.}}{\lambda} I) = 0$$

is called the <u>characteristic equation</u> or the <u>characteristic polynomial</u> of the matrix.

The solutions to this equation are the eigenvalues of the matrix.

Example: Find all eigenvalues of

$$A = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}.$$

Solution: The characteristic equation is

$$\det \begin{pmatrix} -7-\lambda & -6 \\ 9 & 8-\lambda \end{pmatrix} = 0$$

$$(-7-\lambda)(8-\lambda) - (-54) = 0$$

$$\lambda^2 + 7\lambda - 8\lambda - 56 + 54 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda-2)(\lambda+1) = 0$$

$$\left.\begin{aligned} \lambda &= 2 \\ \lambda &= -1 \end{aligned}\right\} \text{These are the eigenvalues of } A.$$

How do we find the corresponding eigenvectors?

Eigenvectors of $\lambda=2$:

We compute Nullspace $(A-2I) = \begin{pmatrix} -7-2 & -6 \\ 9 & 8-2 \end{pmatrix}$

$$= \begin{pmatrix} -9 & -6 \\ 9 & 6 \end{pmatrix}$$

$$\begin{pmatrix} -9 & -6 \\ 9 & 6 \end{pmatrix} \xrightarrow{\text{reducing}} \begin{pmatrix} \boxed{1} & 2/3 \\ 0 & 0 \end{pmatrix}$$

↓ free variable.

Nullspace = set of vectors of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2/3\,y \\ y \end{pmatrix} = y \begin{pmatrix} -2/3 \\ 1 \end{pmatrix}$

Nullspace $(A-2I) = \text{span} \begin{pmatrix} -2/3 \\ 1 \end{pmatrix} = \text{span} \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

eigen vectors corresponding to $\lambda=2$

Eigenvectors for $\lambda=-1$:

Nullspace of $A-(-1)I = \begin{pmatrix} -7+1 & -6 \end{pmatrix}$

Nullspace of $A-(-1)I = \begin{pmatrix} & \\ 9 & 8+1 \end{pmatrix}$

$$= \begin{pmatrix} -6 & -6 \\ 9 & 9 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -6 \\ 9 & 9 \end{pmatrix} \xrightarrow{\text{reduce}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Nullspace $(A-(-1)I)$ = vectors of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Eigenvectors corresponding to $\lambda = -1$ are all multiples of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Ex: Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$$

Solution.

The characteristic equation is

$$\det(A-\lambda I) = 0$$

$$\det(A-\lambda I)=0$$

$$\det\begin{pmatrix}2-\lambda & -3 & \textcircled{1}\\ 1 & -2-\lambda & 1\\ 1 & -3 & 2-\lambda\end{pmatrix}=0$$

$$(2-\lambda)\det\begin{pmatrix}-2-\lambda & 1\\ -3 & 2-\lambda\end{pmatrix}-(-3)\det\begin{pmatrix}1 & 1\\ 1 & 2-\lambda\end{pmatrix}$$

$$+1\cdot\det\begin{pmatrix}1 & -2-\lambda\\ 1 & -3\end{pmatrix}=0.$$

$$(2-\lambda)\Big((-2-\lambda)(2-\lambda)-(-3)\Big)+3(2-\lambda-1)$$

$$+1\cdot\big(-3-(-2-\lambda)\big)=0$$

$$(2-\lambda)\big(\underbrace{-4+\lambda^2+3}_{\lambda^2-1}\big)+3-3\lambda$$

$$-1+\lambda=0.$$

$$2\lambda^2-\lambda^3-2+\lambda+2-2\lambda=0$$

$$-\lambda^3+2\lambda^2-\lambda=0.$$

$$-\lambda(+\lambda^2-2\lambda+1)=0$$

$$-\lambda(\lambda-1)^2=0.$$

$$\left.\begin{array}{l}\lambda = 0 \\ \lambda = 1\end{array}\right\}$$ These are the eigenvalues of the matrix.

To compute the eigenvectors corresponding to these eigenvalues, we need to compute

$$\lambda = 0 \longrightarrow \text{Nullspace}\,(A - 0 \cdot I)$$

$$\lambda = 1 \longrightarrow \text{Nullspace}\,(A - 1 \cdot I).$$

Remarks:

- $\lambda = 0$ is an eigenvalue $\Longleftrightarrow$ A is not invertible.

- If A is an upper-triangular matrix then its eigenvalues are the entries in the diagonal.

$$\det \begin{pmatrix} 3-\lambda & 7 & -2 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda)(2-\lambda)$$

$\lambda = 3$
$\lambda = 2$ eigenvalues

- If A is an $n \times n$ matrix, it has at most n eigenvalues.

at most n eigenvalues.