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What we'll cover today

1] linear maps (general vector spaces)
examples

2] Definitions: (pre)image, kernel ....

3] Rank-nullity theorem (statement)
examples

Linear maps $\mathbb{R}^n \to \mathbb{R}^m$

Ex 1 Set of all polynomials of degree up to 2 with real coefficients P_2

$$f(x) = a_1 + a_2 x + a_3 x^2 \qquad a_1, a_2, a_3 \in \mathbb{R}$$

$$g(x) = b_1 + b_2 x + b_3 x^2 \qquad b_1, b_2, b_3 \in \mathbb{R}$$

$$f(x) + g(x) = (a_1+b_1) + (a_2+b_2)\, x + (a_3+b_3)\, x^2$$

$\forall \; \alpha f(x) \quad \alpha \in \mathbb{R}$ is still $\in P_2$

$$D : P_2 \to P_2 \qquad D = \frac{d}{dx}$$

$$1) \; D(f(x)) = \frac{d}{dx}\left(a_1 + a_2 x + a_3 x^2\right) = a_2 + 2a_3 x$$

$$D(f(x) + g(x)) = D(f(x)) + D(g(x))$$

$$(a_2 + b_2) + 2(a_3 + b_3)x \stackrel{?}{=} a_2 + 2a_3 x + b_2 + 2b_3 x \; ✓$$

$$2) \; D(\alpha f(x)) = \alpha D(f(x))$$

Conditions to be checked to prove that D is a linear map

Of course you can generalise to P_n is the vector space of polynomials of degree up to n with real coefficients

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Ex 2 Consider the space of real 2x2 matrices $\mathbb{R}^{2\times 2}$. A generic element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}$

$$D : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} \qquad D(A) = A - A^T$$ is a linear map

$$D(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ -(b-c) & 0 \end{pmatrix} \leftarrow$$

1) $D(A_1 + A_2) \overset{?}{=} D(A_1) + D(A_2)$

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

2) $D(\underset{?}{\alpha} A) = \alpha D(A)$

$\alpha \in \mathbb{R}$

Comments

- P_n has dimension $n+1$

$\underline{v}_1 = 1, \quad \underline{v}_2 = x, \quad \underline{x}_3 = x^2, \ldots, \quad v_{n+1} = x^n$ → span P_n, lin. ind.

$$\sum_{i=1}^{n+1} a_i x^{i-1} = \sum_{i=1}^{n+1} a_i \underline{v}_i$$

$\{\underline{v}_i\}$ is a basis $\Rightarrow \dim(P_n) = n+1$

- The vector space $\mathbb{R}^{n \times m}$ has dim. $n \cdot m$

$\mathbb{R}^{2\times 2}$ has dim. 4

- $D = \frac{d}{dx}$ on polynomials

$D: P_2 \to P_1 \subset P_2$

$D: A - A^T$ $\quad \mathbb{R}^{2\times 2} \to$ set of 2x2 antisymm. matrices $\subset \mathbb{R}^{2\times 2}$

$\downarrow$

dimension 1 $\quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

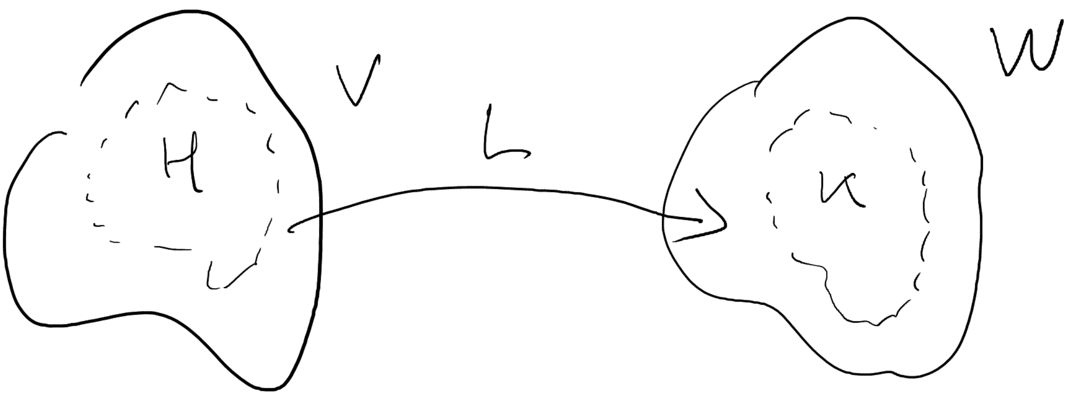
2] Let $L : V \to W$ ← vector spaces

↑

linear map

$H \subseteq V$ $\quad$ H is a vector subspace of V

$K \subseteq W$ $\quad$ K " " " " " W



def: the pre image of K under L

$$L^{-1}(K) = \{ \underline{v} \in V : L(v) \in K \}$$

theorem $L^{-1}(K)$ is a vector subspace of V

$\underline{v}_1, \underline{v}_2 \in L^{-1}(K)$ $\quad a_1 \underline{v}_1 + a_2 \underline{v}_2 \in L^{-1}(K)$ $\quad a_i \in \mathbb{R}$

$$L(a_1 \underline{v}_1 + a_2 \underline{v}_2) \in K$$

$\downarrow$ linear map

$$a_1 L_1(\underline{v}_1) + a_2 L(\underline{v}_2) \in K \quad \checkmark$$

$\in K \qquad \in K$

- def $K = \{\underline{0}\} \subset W$ $\qquad$ kernel

$$L^{-1}(\{\underline{0}\}) = \ker(L)$$

$$\ker(L) = \{ \underline{v} \in V : L(\underline{v}) = \underline{0} \}$$

def The image of H under L

$$L(H) = \{ \underline{w} \in W : \underline{w} = L(\underline{v}) \text{ for some } v \in H \}$$

$$L(H) \subset W$$

Theor. $L(H)$ is a vector subspace of W

<u>Def</u> range of the linear map L
image " " " " "

$$L(V) = \mathrm{im}(L)$$

3] Definitions

integer numbers $0, 1, 2, 3 \ldots$

$$\dim(\ker(L)) = \mathrm{nul}(L)$$ nullity of L

$$\dim(\mathrm{im}(L)) = \mathrm{rank}(L)$$ rank of L

<u>Rank-Nullity theorem</u>

$$\boxed{\mathrm{rank}(L) + \mathrm{nul}(L) = \dim(V)}$$