Lecture Nov 16

16 November 2023 09:07

Change-of-coordinates formula

Suppose V is a vector space, and $B_1 = \{v_1, v_2, \ldots, v_n\}$ and B_2 are bases of V.

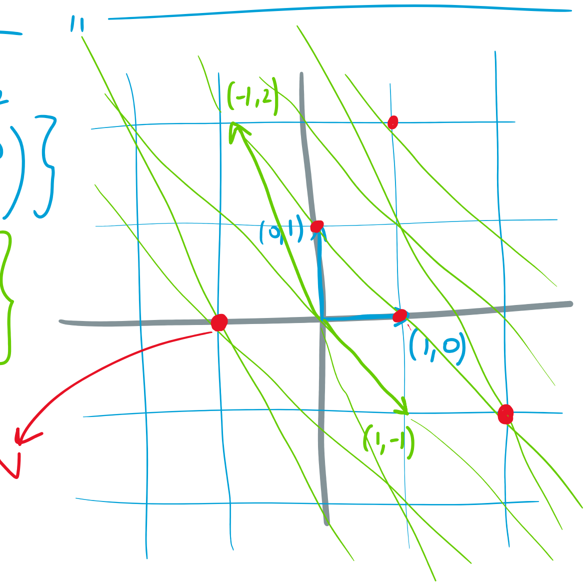
Then

$$[v]_{B_2} = \begin{bmatrix} | & | & & | \\ [v_1]_{B_2} & [v_2]_{B_2} & \cdots & [v_n]_{B_2} \\ | & | & & | \end{bmatrix} \cdot [v]_{B_1}$$

Ex: $V = \mathbb{R}^2$.

$$B_1 = \left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$



$$[v]_{B_2} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$[v]_{B_1} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Change-of-coordinates formula:

$$[v]_{B_2} = \begin{bmatrix} | & | \\ [v_1]_{B_2} & [v_2]_{B_2} \\ | & | \end{bmatrix} [v]_{B_1}$$

For every v

$$[v]_{B_2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} [v]_{B_1}$$

In our example

$$\begin{pmatrix}-2\\-1\end{pmatrix} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} \cdot \begin{pmatrix}-1\\0\end{pmatrix}$$

↳ change-of-coordinates matrix

Matrix associated to a linear transformation

Suppose $T: V \to W$ is a linear transformation.
Can we represent T by a matrix?
Choose a basis B_1 for V
and a basis B_2 for W.

<u>Ex</u>: Suppose $T: \underset{\text{basis } B_1}{\underset{\uparrow}{P_1}} \to \underset{\text{basis } B_2}{\underset{\uparrow}{P_2}}$ is a linear transformation.

Suppose $[T(1)]_{B_2} = \begin{pmatrix}1\\3\\-2\end{pmatrix}$

$$[T(x)]_{B_2} = \begin{pmatrix}0\\-1\\2\end{pmatrix}$$

<u>Q</u>: Can we know

$$[T(1+x)]_{B_2} = [T(1)+T(x)]_{B_2}$$

$$= [T(1)]_{B_2} + [T(x)]_{B_2}$$

$$= \begin{pmatrix}1\\3\\-2\end{pmatrix} + \begin{pmatrix}0\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The matrix representing T:

- $$[T(v)]_{B_2} = \begin{pmatrix} | & | & & | \\ [T(v_1)]_{B_2} & [T(v_2)]_{B_2} & \cdots & [T(v_n)]_{B_2} \\ | & | & & | \end{pmatrix} [v]_{B_1}$$

Ex: Consider $T: P_3 \longrightarrow P_2$. defined as

$$T(p(x)) = p'(1) \cdot x + p(1).$$

Let's choose bases

$B_1 = \{1, x, x^2, x^3\}$ basis for P_3

$B_2 = \{1, x, x^2\}$ basis for P_2.

$T(1) = 0x + 1$

$T(x) = 1x + 1$

$T(x^2) = 2x + 1$

$T(x^3) = 3x + 1$

$$[T(p(x))]_{B_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} [p(x)]_{B_1}$$

$\downarrow$

matrix representing T in the bases B_1, B_2.

For instance, if we wanted to know

$$T(3 + 7x - 5x^2 - 3x^3)$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$\left[3 + 7x - 5x^2 - 3x^3\right]_{B_1} = \begin{pmatrix} 3 \\ 7 \\ -5 \\ -3 \end{pmatrix}$$

Our formula says:

$$\left[T(p(x))\right]_{B_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \\ -5 \\ -3 \end{pmatrix}$$

$$\left[T(p(x))\right]_{B_2} = \begin{pmatrix} 2 \\ -12 \\ 0 \end{pmatrix}$$

$$\boxed{T(p(x)) = 2 \cdot 1 - 12x + 0x^2}$$

You can use this matrix A to compute the kernel and the image of the transformation T.

kernel T $\rightsquigarrow$ nullspace A

image T $\rightsquigarrow$ column space A.