Linear transformations, image, kernel

- $T: V \to W$ a function between two vector spaces is a linear transformation if
  - $T(v+w) = T(v) + T(w)$
  - $T(cv) = cT(v)$

Ex: P_2 is the vector space of polynomials of degree at most 2.

$x^2 - 7x + 5 \in P_2$

$T: P_2 \to P_2$ defined as $T(p(x)) = p'(x)$.

is a linear transformation.

Ex: $M_{2\times2}$ is the vector space of 2×2 matrices. The function

$T: M_{2\times2} \to M_{2\times2}$ defined by

$$T(A) = A - A^T.$$

For instance $T\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} - \begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$

$$T\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} - \begin{pmatrix}1 & 3\\ 2 & 4\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$T\begin{pmatrix}2 & 3\\ 5 & 6\end{pmatrix} = \begin{pmatrix}2 & 3\\ 5 & 6\end{pmatrix} - \begin{pmatrix}2 & 5\\ 3 & 6\end{pmatrix} = \begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}$$

In general

$$T(A+B) = (A+B) - (A+B)^T =$$

$$= A + B - A^T - B^T = \underline{A - A^T} + \underline{B - B^T}$$

$$= T(A) + T(B)$$

We also have $T(cA) = cT(A)$

This T is a <u>linear transformation</u>.

<u>Image and Kernel</u>. Suppose $T: V \to W$ is a linear transformation.

$$\text{Image}(T) = \{T(v) \mid v \in V\}.$$

$$\text{Kernel}(T) = \{v \in V \mid T(v) = 0\}.$$

- Image (T) is a subspace of W.
- Kernel (T) is a subspace of V.

<u>Ex</u>: $T: P_2 \longrightarrow P_2$

$$T(p(x)) = p'(x).$$

Image (T) = polynomials of degree at most 1.

Kernel (T) = constant polynomials.

Let's compute bases for these subspaces:

T … (T): $\{1, x\}$ $\leftarrow \boxed{\dim = 2}$

Lets compute

- Basis for Image(T): $\{1, x\} \leftarrow \boxed{\dim = 2}$
- Basis for Kernel (T): $\{1\} \leftarrow \boxed{\dim = 1}$

<u>Ex</u>: $T: M_{2\times 2} \longrightarrow M_{2\times 2}$

$$T(A) = A - A^T.$$

Image (T) = subspace of all skew-symmetric matrices.

Kernel (T) = subspace of all symmetric matrices.

- Basis for Image (T) = $\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\} \rightarrow \boxed{\dim = 1}$
- Basis for Kernel (T) = $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \rightarrow \boxed{\dim = 3}$

a general symm. matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$$

$$= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

a general skew-symm. matrix

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = b \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

<u>Rank-nullity theorem</u>:

$$\boxed{\underbrace{\dim\left(\mathrm{Image}(T)\right)}_{\text{called rank of } T} + \underbrace{\dim\left(\mathrm{Kernel}(T)\right)}_{\text{called nullity of } T} = \underbrace{\dim(V)}_{\text{domain of } T}.}$$

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Coordinatisation of a vector space.

Recall: If $\mathcal{B} = \{v_1, \dots, v_n\}$ is a collection of vectors that forms a basis in a vector space V, then any vector of V can be written uniquely as

$$v = \underset{\downarrow}{a_1} v_1 + \underset{\downarrow}{a_2} v_2 + \dots + \underset{\downarrow}{a_n} v_n$$

called the coordinates of v in the basis $\mathcal{B}$.

$$\underset{\downarrow}{[v]_{\mathcal{B}}} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

vector of coordinates of v in the basis $\mathcal{B}$.

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Ex: V = 2×2 symmetric matrices

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

What are the coordinates of $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ in the basis $\mathcal{B}$?

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\left[\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Exercise: $V = P_1 \leftarrow$ polynomials of degree at most 1.

Fix $\mathcal{B} = \{x-3, -x+2\}$.

Q. What are the coordinates of $p(x) = 7x - 5$ in the basis $\mathcal{B}$?

$$7x - 5 = a(x-3) + b(-x+2)$$

$$7 = a - b$$
$$-5 = -3a + 2b$$

$$\left(\begin{array}{cc|c} 1 & -1 & 7 \\ -3 & 2 & -5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 7 \\ 0 & -1 & 16 \end{array}\right)$$

$$\boxed{\begin{array}{l} b = -16 \\ a = -9 \end{array}}$$

$$[7x-5]_{\mathcal{B}} = \begin{pmatrix} -9 \\ -16 \end{pmatrix}.$$

Remark:

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Questions about elements of the vector space V can be translated into questions about their coordinate vectors.

Ex: V = 2x2 symm. matrices

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The question

- Are the matrices $\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 7 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ linearly dependent or independent?

is equivalent to

- Are their coordinate vectors $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ linearly dependent or independent?

Change of coordinates:

Ex: $P_1 \leftarrow$ polynomials of degree at most 1.

Suppose $\mathcal{B}_1 = \{ p_1, p_2 \}$ is a basis

$\mathcal{B}_2 = \{ q_1, q_2 \}$ is another basis.

If we know $[f(x)]_{B_1}$, how do we know $[f(x)]_{B_2}$?

Answer: We just need to know how to change the coordinates for the elements in the basis B_1.

Ex: Suppose
$$p_1 = 3q_1 - q_2$$
$$p_2 = 5q_1 - 3q_2 .$$
Example

If we know
$[f]_{B_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, what are $[f]_{B_2}$?

$$f = 2\,p_1 + 1\,p_2$$
$$= 2\,(3q_1 - q_2) + 1 \cdot (5q_1 - 3q_2)$$
$$= (2 \cdot 3 + 1 \cdot 5)\,q_1 + (2 \cdot (-1) + 1 \cdot (-3))\,q_2$$

$$[f]_{B_2} = \begin{pmatrix} 2 \cdot 3 + 1 \cdot 5 \\ 2 \cdot (-1) + 1 \cdot (-3) \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} | & | \\ [p_1]_{B_2} & [p_2]_{B_2} \\ | & | \end{pmatrix} [f]_{B_1}.$$

Theorem (Change-of-coordinates formula).

Suppose V is a vector space and B_1, B_2 are bases for V. Then

$B_1 = \{v_1, \dots, v_n\}$

for any element $v \in V$

$$[v]_{B_2} = \begin{bmatrix} | & | & & | \\ [v_1]_{B_2} & [v_2]_{B_2} & \cdots & [v_n]_{B_2} \\ | & | & & | \end{bmatrix} \cdot [v]_{B_1}$$

change-of-coordinates matrix from B_1 to B_2.