Lecture Dec 7

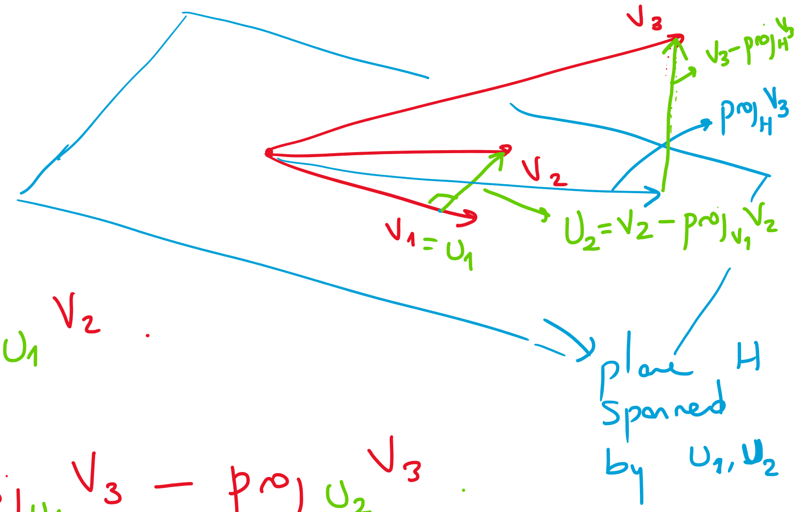
07 December 2023 09:09

Gram-Schmidt process:

It constructs an orthogonal basis for a subspace starting with any basis.

Let's say we start with basis $\mathcal{B} = \{v_1, \dots, v_m\}$.

Gram-Schmidt:



$$u_1 = v_1$$

$$u_2 = v_2 - \text{proj}_{u_1} v_2.$$

$$u_3 = v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3.$$

In general:

$$u_k = v_k - \text{proj}_{u_1} v_k - \text{proj}_{u_2} v_k - \dots - \text{proj}_{u_{k-1}} v_k$$

$$= v_k - \frac{v_k \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_k \cdot u_2}{u_2 \cdot u_2} u_2 - \dots - \frac{v_k \cdot u_{k-1}}{u_{k-1} \cdot u_{k-1}} u_{k-1}$$

Example: Find an orthogonal basis for the subspace H with basis

$$\left\{ \overset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \end{pmatrix}}, \overset{v_3}{\begin{pmatrix} 0 \\ 0 \\ 2 \\ \vdots \end{pmatrix}} \right\}$$

H with basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix} \right\}$

Solution: Use Gram-Schmidt process:

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} - \frac{4}{4} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} - \frac{8}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

The orthogonal basis produced is

The orthogonal ...

$$\mathcal{B}' = \{ u_1, u_2, u_3 \} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Least-squares approximations:

In many applications, we are interested in finding a solution to a system

$$A \vec{x} = \vec{b}$$

which turns out to be inconsistent.

In these cases, we can find the "best approximation" to an exact solution.

Definition: A least-squares approximation of the system

$$A \vec{x} = \vec{b} \qquad A \in \mathbb{R}^{m \times n}$$

is a vector $v \in \mathbb{R}^n$ such that the distance

$$\operatorname{dist}(Av, b)$$

is as small as possible.

In other words

$$\text{dist}(Av, b) \leq \text{dist}(Ax, b) \text{ for any } x \in \mathbb{R}^n.$$

$\text{dist}(Av, b) = \|Av - b\|$

v: least-squares solution

The vector v is also called a least-squares solution.

"

Example: Suppose we want to find a solution to

$$\left.\begin{aligned} 4x + 0y &= 2 \\ 0x + 2y &= 0 \\ x + y &= 11 \end{aligned}\right\} \text{Inconsistent}$$

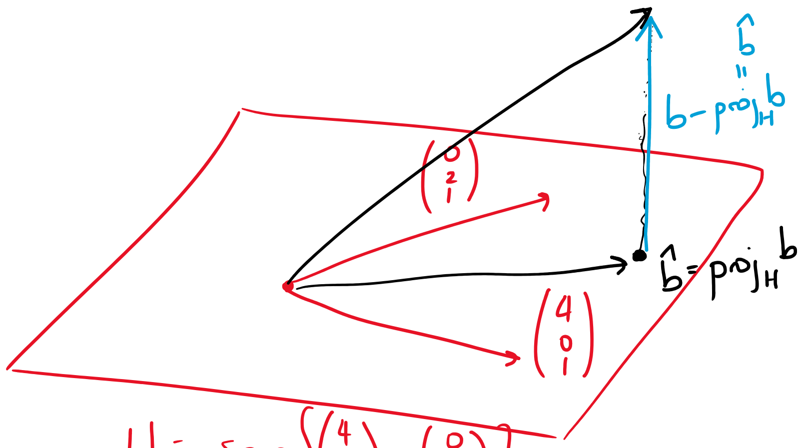
In matrix form:

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

or

$$x \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

$b = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$



$H = \mathrm{span}\left\{ \begin{pmatrix}4\\0\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix} \right\}$

We want to find a vector v such that $Av = \hat{b}$

$$b - \hat{b} = b - Av \quad \leftarrow \text{blue vector}$$

$$\begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \underbrace{(b - Av)}_{\text{blue vector}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that:

$$A^T(b - Av) = 0$$

$$\Rightarrow \boxed{A^T A v = A^T b}$$

normal equations

Solving this would give the least squares

Solving this would give you the least-squares approximations.