Lecture Dec 4

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Recall:

- An orthogonal set of vectors $\{v_1, v_2, \ldots, v_k\}$ in $\mathbb{R}^n$ is a set of vectors that are pairwise orthogonal.
- Fact: Orthogonal set $\Longrightarrow$ Linearly independent set
- An orthogonal basis is a basis for a subspace that is also an orthogonal set.

Example: Find an orthogonal basis for the subspace

$$x + 2y - z = 0$$

Solution: We just need to find two linearly independent vectors in that plane:

For instance $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \right\}$.

- Finding coordinates with respect to an orthogonal basis

Theorem: If $B = \{v_1, \ldots, v_k\}$ is an orthogonal basis for a subspace H then the coordinates of a vector v are:

$$\begin{pmatrix} \frac{v \cdot v_1}{v_1 \cdot v_1} \\ \vdots \end{pmatrix}$$

v are :

$$[v]_B = \begin{pmatrix} \dfrac{v \cdot v_1}{v_1 \cdot v_1} \\[2mm] \dfrac{v \cdot v_2}{v_2 \cdot v_2} \\ \vdots \\ \dfrac{v \cdot v_k}{v_k \cdot v_k} \end{pmatrix}$$

coordinates of v with respect to B

Example: Consider the orthogonal basis for $\mathbb{R}^3$

$$B = \left\{ \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix} \right\}.$$

$\uparrow v_1 \qquad \uparrow v_2 \qquad \uparrow v_3$

Find the coordinates of $v = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ with respect to B.

Solution: Because this is an orthogonal basis, we can find the coordinates as:

$$[v]_B = \begin{pmatrix} \dfrac{v \cdot v_1}{v_1 \cdot v_1} \\[2mm] \dfrac{v \cdot v_2}{v_2 \cdot v_2} \\[2mm] \dfrac{v \cdot v_3}{v_3 \cdot v_3} \end{pmatrix} = \begin{pmatrix} \dfrac{11}{11} \\[2mm] \dfrac{-12}{6} \\[2mm] \dfrac{-66}{66} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$$

Proof of the theorem:

The coordinates of v with respect to B are the coefficients we need to use to write

$$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_k v_k$$

Taking dot product with v_1 we get

$$v \cdot v_1 = a_1 \, v_1 \cdot v_1 + a_2 \, \cancelto{0}{v_1 \cdot v_2} + a_3 \, \cancelto{0}{v_1 \cdot v_3} + \dots + a_k \, \cancelto{0}{v_1 \cdot v_k}$$

$$v_1 \cdot v = a_1 \, v_1 \cdot v_1 + a_2 \, \cancel{v_1 \cdot v_2}^{\,0} + a_3 \, \cancel{v_1 \cdot v_3}^{\,0} + \dots + a_k \, \cancel{v_1 \cdot v_k}^{\,0}$$

$$\frac{v_1 \cdot v}{v_1 \cdot v_1} = a_1$$

A similar argument, taking dot product with other v_is, we get

$$\frac{v_2 \cdot v}{v_2 \cdot v_2} = a_2 \qquad \dots \qquad \frac{v_k \cdot v}{v_k \cdot v_k} = a_k .$$

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Orthonormal sets:

Definition: An <u>orthonormal set</u> is an orthogonal set of vectors all of length equal to 1.

Example: We can turn the orthogonal set in the previous example into an orthonormal set by dividing the vectors by their length:

$$\left\{ \begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{pmatrix} \right\}$$

Remark: If $\mathcal{B}$ is an orthonormal basis, the formula above for the coordinates of a vector becomes even simpler:

$$[v] = \begin{pmatrix} v \cdot v_1 \\ v \cdot v_2 \end{pmatrix}$$

$$[v]_B = \begin{pmatrix} v \cdot v_1 \\ v \cdot v_2 \\ \vdots \\ v \cdot v_k \end{pmatrix}$$

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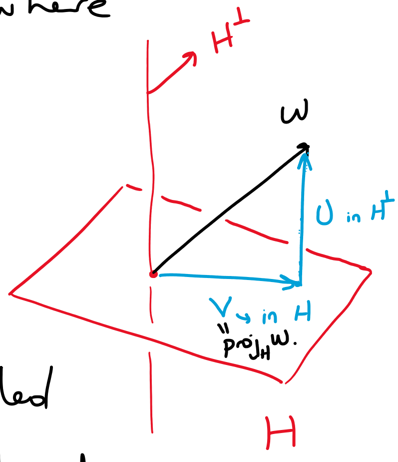
Orthogonal projections:

Proposition: Suppose H is a subspace of $\mathbb{R}^n$. Any vector w in $\mathbb{R}^n$ can be decomposed as

$$w = v + u \quad \text{where}$$

v is in H

and u is in $H^\perp$



The vector v is called the <u>projection of w onto H</u> denoted by $\text{proj}_H w$.

If $B = \{v_1, \dots, v_k\}$ is an orthogonal basis for H then

$$\boxed{\text{proj}_H w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \cdots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k}$$

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Example: Find the projection of $w = \begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}$ onto the plane $x + 2y - z = 0$.

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$w = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ onto

<u>Solution:</u> We computed above an orthogonal basis for the plane

$$B = \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}} \right\}.$$

The projection of w onto the plane is

$$\mathrm{proj}_H w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \frac{3}{2} v_1 + \frac{5}{3} v_2$$

$$= \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{5}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 19/6 \\ -5/3 \\ -1/6 \end{pmatrix}$$

<u>Proof of the proposition:</u>

Suppose w is in $\mathbb{R}^n$ and $B = \{v_1, \dots, v_k\}$ is an orthogonal basis for H.

Let $$v = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \dots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k$$

This vector v is in H, because it is a combination of the v_i s.

We need to check that

$w - v$ is in $H^\perp$.

To check this, we need to check

To check this, we need to check that $w - v$ is orthogonal to all the v_i

$$(w-v)\cdot v_1 = w\cdot v_1 - v\cdot v_1$$

$$= w\cdot v_1 - \left(\frac{w\cdot v_1}{v_1\cdot v_1}v_1 + \dots + \frac{w\cdot v_k}{v_k\cdot v_k}v_k\right)\cdot v_1$$

$$= w\cdot v_1 - \frac{w\cdot v_1}{\cancel{v_1\cdot v_1}}\cancel{v_1\cdot v_1} - \frac{w\cdot v_2}{v_2\cdot v_2}\overset{0}{\cancel{v_2\cdot v_1}} - \overset{0}{\cancel{\cdots}} - \frac{w\cdot v_k}{v_k\cdot v_k}\overset{0}{\cancel{v_k\cdot v_1}}$$

$$= w\cdot v_1 - w\cdot v_1$$

$$= 0 .$$

This shows that $w - v$ is orthogonal to all the v_i, so it is in $H^\perp$. $\square$

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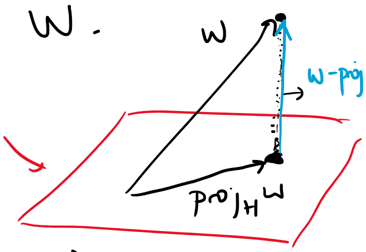
<u>Proposition</u>: The projection $\mathrm{proj}_H w$ is the vector in H that is closest to the vector w.



In other words,

$$\mathrm{dist}(w, \mathrm{proj}_H w) \leq \mathrm{dist}(w, v)$$

for any $v \in H$.

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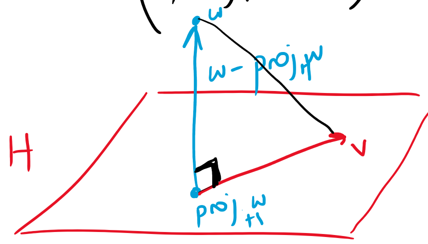
<u>Proof</u>: $w - \mathrm{proj}_H w$ is orthogonal to any

vector v in H.

Using the Pythagorean Theorem, we get

$$\text{dist}(v,w)^2 = \text{dist}(v, \text{proj}_H w)^2 + \text{dist}(\text{proj}_H w, w)^2$$



for any v in H

So

$$\text{dist}(v,w)^2 \geq \text{dist}(\text{proj}_H w, w)^2.$$

So $\text{dist}(v,w) \geq \text{dist}(\text{proj}_H w, w)$ ∎