APPLIED LINEAR ALGEBRA

WEEK 12 - LESSON 1

1801 Italian astronomer observes Ceres (dwarf planet)

position in sky.

then sun appeared → lost track.

where would it be after transit behind sun?

Orbit depended on 3 parameters x, y, z

To find x, y, z need to solve some linear equations One for each observation ②

$$\left.\begin{array}{l} 3x - 2y + 1z = 7 \\ 4x + y - 7z = 5 \\ \vdots \\ \vdots \end{array}\right\}$$

1 eqn per observation

Much more eqns than unknowns

Systems like this do not tend to have solution (e.g. noisiness in observations)

What to do?

This is an incompatible linear system

$$A\vec{x} = \vec{b}$$

Need some maths (Gauss)

<u>Idea:</u> Get x such that

Ax is as close as possible to b.

i.e.

$$Ax - b$$

as small as possible.

i.e.

$$\|Ax - b\|$$

is as small as possible.

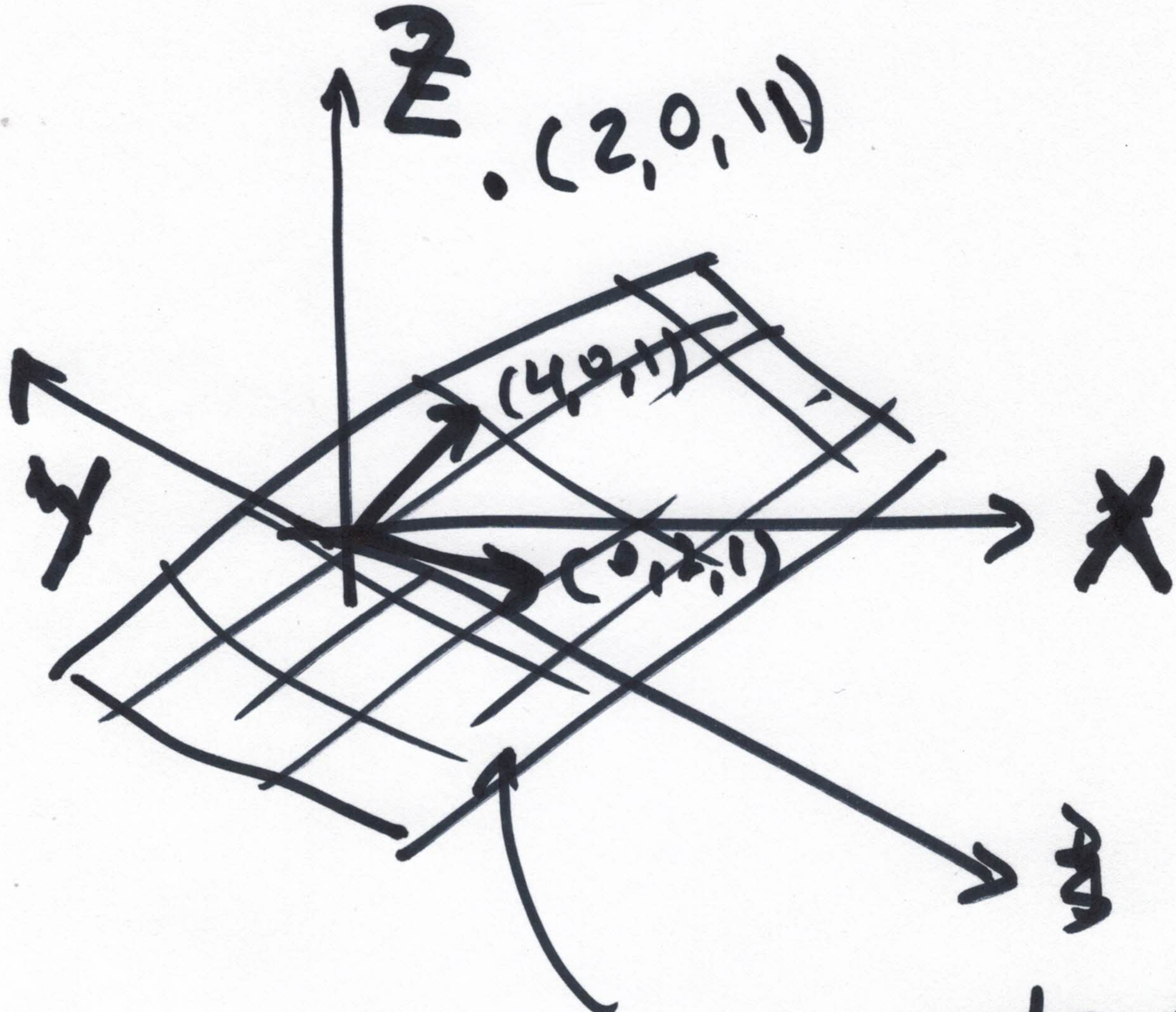
Such an x will be called the <u>least squares solution</u> of $Ax \doteq b$ (even though it is not a solution) 4

How do we find such an x?

Let us look at a picture In the case

$$\begin{cases} 4x + 0y = 2 \\ 0x + 2y = 0 \\ \underbrace{x + y}_{\substack{\text{3 dim vector} \\ \text{depending on } x,y}} = \underbrace{11}_{\text{3 dim vector}} \end{cases}$$ (incompatible)

How do we draw this?



here reside all

$$(4x+0y,\ 0x+2y,\ x+y)$$

$$\cancel{= 4(1,0,1) +}$$

$$= x(4,0,1) + y(0,2,1)$$

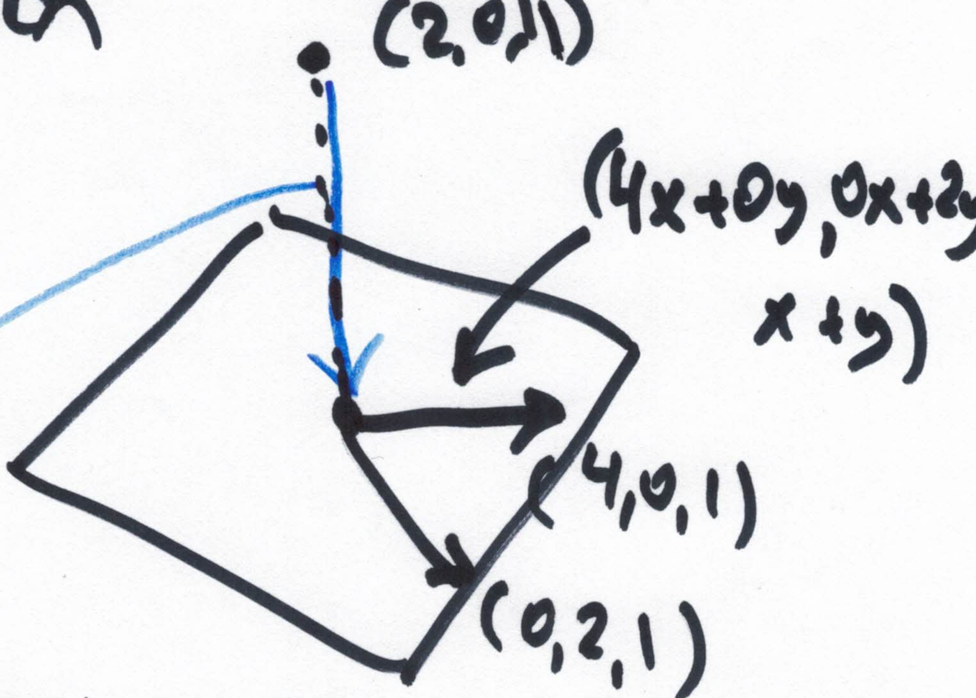
Need to find (x,y) such that

$$(4x+0y,\ 0x+2y,\ x+y)$$

is as close as possible to

$$(2,\ 0,\ 11)$$

Answer : orthogonal projection



just need to demand

$$(4x+0y,\ 0x+2y,\ x+y)-(2,0,1)$$

perpendicular to $(4,0,1)$ & $(0,2,1)$

$$\langle (4x+0y,\ 0x+2y,\ x+y)-(2,0,1),\ (4,0,1)\rangle = 0$$

$$\big((4x+0y,\ 0x+2y,\ x+y)-(2,0,1)\big)\cdot(0,2,1) = 0$$

This is 2 eqns in x, y ✓

$$\begin{cases} 16x + x + y = 8 + 11 \\ 4y + x + y = 11 \end{cases}$$ 7

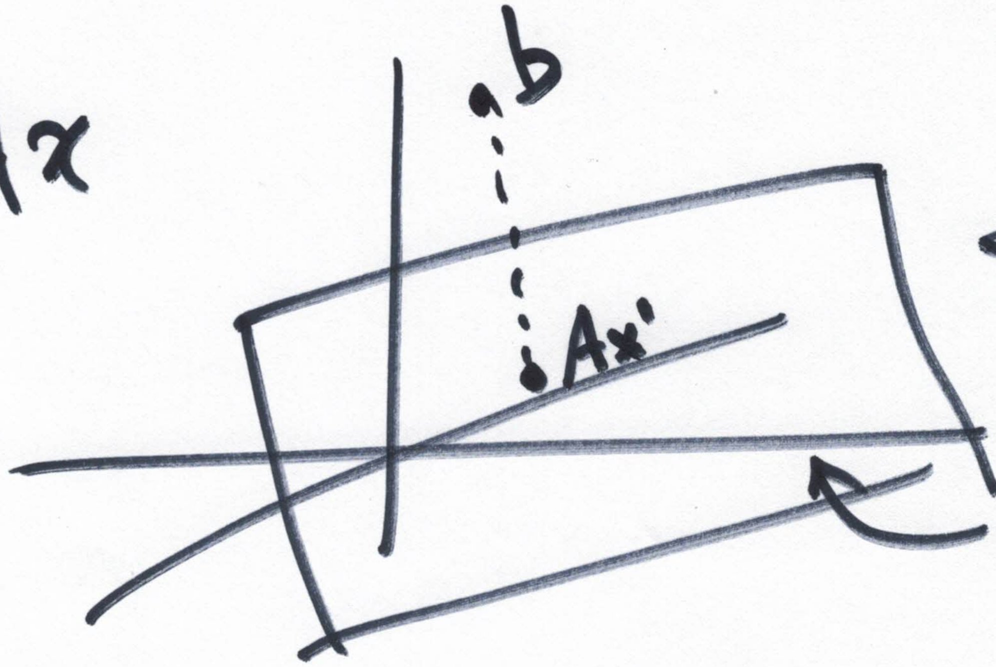
$$\begin{cases} 17x + y = 19 \\ x + 5y = 11 \end{cases}$$

~~(Some mistake in arithmetic → homework)~~

Solution is $(x, y) = (1, 2)$

How does this work more abstractly?
(need this if numbers or # of variables gets big)

Ax



values of Ax
vector

find x s.t. Ax as close as possible to b.

$\Rightarrow$ projection of b on this subspace

$\Rightarrow$ $Ax' - b$ is perpendicular to all vectors in $\cancel{Ax}$ span of columns of A.

$\Rightarrow$ Scalar product of columns of A with $Ax' - b$ is zero.

$$A^T(Ax'-b)=0$$

$$A^TAx'=A^Tb$$

<u>**Remark**</u> If A is arbitrary (~~not square~~)

the A^TA is always square

$$A=\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \qquad A^T=\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$$

$$A^TA=\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}\begin{pmatrix} a & d \\ c & d \\ e & f \end{pmatrix}$$

$$=\begin{pmatrix} \vdots & \vdots \end{pmatrix} \quad \text{square.}$$

Therefore

$$A^TAx=A^Tb$$

is a $n\times n$ system which we can solve.

Conclusion If $Ax = b$

is a system with more eqns than unknowns then it is usually incompatible.
In order to find x s.t. Ax is as close possible to b we must solve

$$A^T A x = A^T b$$

(normal equations)

This is how the orbit of Ceres was found.

Example Let us redo

using normal equations

$$\begin{cases} 4x + 0y = 2 \\ 0x + 2y = 0 \\ x + y = 11 \end{cases}$$

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 11 \end{pmatrix} = \begin{pmatrix} 19 \\ 11 \end{pmatrix}$$

Normal eqns

$$\begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 \\ 11 \end{pmatrix}$$

$\rightsquigarrow$ Solution is $(x, y) = (1, 2)$

<u>Remark</u> If $A^T A$ is invertible

then $$x = (A^T A)^{-1} A^T b$$

this happens when A has maximum rank

Remark Incompatible systems like the ones above are sometimes called "over determined"

In applications, where observations are noisy, you do not want 3 eqns for 3 unknowns

Because if observations are a bit wrong then 3 eqns will be a bit wrong & solutions could be very wrong.

$\Rightarrow$ Want many eqns for 3 unknowns.

Applications of linear algebra

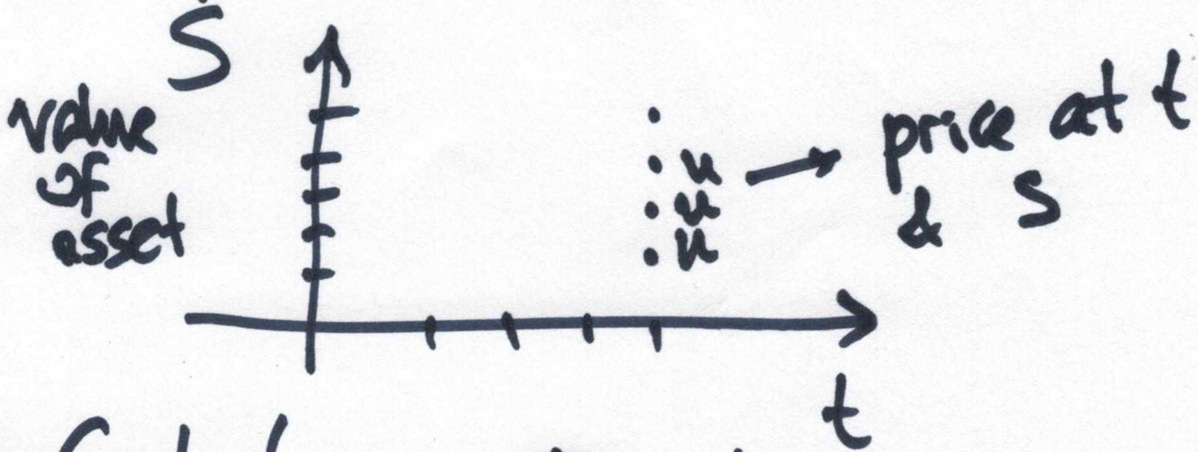
→ linear algebra is everywhere

Some examples

<u>In Finance</u> in pricing derivatives need to solve Partial Differential Equations ($\frac{du}{dt}$, $\frac{du}{dS}$, ...

Often cannot find an analytical solution → Need to approximate

end up discretizing space & time

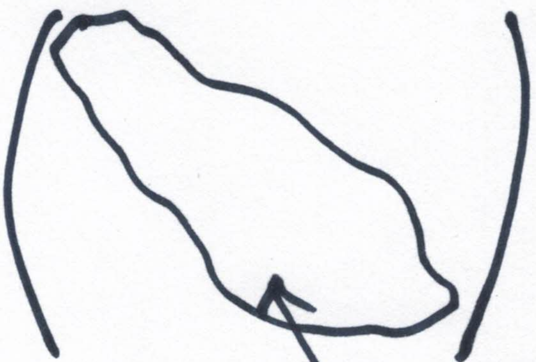


⇒ Get linear Equations

200 unknowns $\times$ 200 eqns

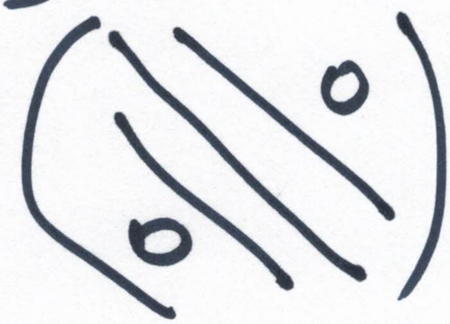
$\times$ 200

They are "sparse linear equations"



non zero only in some region.

tridiagonal systems



<u>In maths</u> Fibonacci sequence

$$x_0 = 1, \quad x_1 = 1, \quad x_n = x_{n-1} + x_{n-2}$$

$$x_2 = 2, \; x_3 = 3, \; x_4 = 5, \; x_5 = 8, \ldots$$

How do these numbers grow?

Nice way to solve this:

Can we convert this to a recurrence with 1 step?

let us use vectors:

$$\cancel{\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \Big(}$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

In particular

$$\begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

So

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we can calculate $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$ we can get closed form for x_n.

To do this, we diagonalise $\begin{pmatrix}1&1\\1&0\end{pmatrix}$

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$$\begin{pmatrix}1&1\\1&0\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}^{-1}\begin{pmatrix}\cdot&0\\0&\cdot\end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$

eigen values   eigen vectors

Eigenvalues

$$0 = \begin{vmatrix}1-x & 1\\ 1 & -x\end{vmatrix} = x^2 - x - 1 \rightsquigarrow x^{-1} = 1 + x$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} = \begin{matrix} 1.61\ldots \; (=\sigma) \\ 0.61\ldots \; (\text{"}\sigma^{-1}\text{"}) \end{matrix}$$

Golden ratio : σ

$$\begin{pmatrix}1&1\\1&0\end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}^{-1}\begin{pmatrix}1.61 & \\ & 0.61\end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$

Eigen vectors

$$\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\sigma x\\ \sigma y\end{pmatrix}$$

→ $\exists$ non zero solution

→ only 1 indep eqn.

$\hookrightarrow x + y = \sigma x$

$(1, \sigma - 1)$

other

$(1, \sigma^{-1} - 1)$

So

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sigma-1 & \sigma^{-1}-1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma^n & \\ & (\sigma^{-1})^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \sigma-1 & \sigma^{-1}-1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

& doing "the algebra"

we get $x_n = \ldots \sigma^n + \ldots + \sigma^{-n}$
{homework}

where $\sigma = 1.61$ golden ratio
$\sigma^{-1} = 0.61$

These same ideas work for any recurrence

$$\left| \begin{array}{l} x_{n+1} = 3x_n - 4x_{n-1} + 7x_{n-2} \\ x_0 = x_1 = x_2 = 3 \end{array} \right.$$

(here will use vectors of size 3)

Newton Raphson

$f(x) = 3$

$\frac{\sin x}{x} = 3$

$x_0 = 1$

$x_{n+1} = x_n - \frac{f(x_n)}{f''(x_n)}$

$\rightarrow$ solution.