Chapter 5

Rotation Curves

5.1 Circular Velocities and Rotation Curves

The *circular velocity* v_{circ} is the velocity that a star in a galaxy must have to maintain a circular orbit at a specified distance from the centre, on the assumption that the gravitational potential is symmetric about the centre of the orbit. In the case of the disc of a spiral galaxy (which has an axisymmetric potential), the circular velocity is the orbital velocity of a star moving in a circular path in the plane of the disc. If the absolute value of the acceleration is g, for circular velocity we have $g = v_{circ}^2/R$ where R is the radius of the orbit (with R a constant for the circular orbit). Therefore, $\partial\Phi/\partial R = v_{circ}^2/R$, assuming symmetry.

The *rotation curve* is the function $v_{circ}(R)$ for a galaxy. If $v_{circ}(R)$ can be measured over a range of R, it will provide very important information about the gravitational potential. This in turn gives fundamental information about the mass distribution in the galaxy, including dark matter.

We can go further in cases of spherical symmetry. Spherical symmetry means that the gravitational acceleration at a distance R from the centre of the galaxy is simply $GM(R)/R^2$, where $M(R)$ is the mass interior to the radius R. In this case,

$$\frac{v_{circ}^2}{R} = \frac{GM(R)}{R^2} \qquad \text{and therefore,} \qquad v_{circ} = \sqrt{\frac{GM(R)}{R}} \; . \tag{5.1}$$

If we can assume spherical symmetry, we can estimate the mass inside a radial distance R by inverting Equation 5.1 to give

$$M(R) = \frac{v_{circ}^2 \, R}{G} \; , \tag{5.2}$$

and can do so as a function of radius. This is a very powerful result which is capable of telling us important information about mass distribution in galaxies, provided that we have spherical symmetry. However, we must use a more sophisticated analysis for the general case where we do not have spherical symmetry. The more general case of axisymmetry is considered in Section 5.3.

5.2 Observations

Gas and young stars in the disc of a spiral galaxy will move on nearly circular orbits (especially if the potential is truly axisymmetric). Therefore if the bulk rotational velocity v_{rot} of gas or young stars can be measured, it provides $v_{circ}^2 = R\,\partial\Phi/\partial R$. Old stars should be avoided: old stars have a greater velocity dispersion around their mean orbital motion and their bulk rotational velocity will be slightly smaller than the circular velocity.

Spectroscopic radial velocities can be used to determine the rotational velocities of spiral galaxies provided that the galaxies are inclined to the line of sight. The analysis is impossible for face-on spiral discs, but inclined spirals can be used readily. The velocity v_{rot} is related to the velocity component v_l along the line of sight (corrected for the bulk motion of the galaxy) by $v_l = v_{rot} \cos i$ where i is the inclination angle of the disc of the galaxy to the line of sight (defined so that $i = 90°$ for a face-on disc). Placing a spectroscopic slit along the major axis of the elongated image of the disc on the sky provides the rotation curve from optical observations. Radio observations of the 21 cm line of neutral hydrogen at a number of positions on the disc of the galaxy can also provide rotation curves, and often to larger radii than optical ones.

For example, in our Galaxy the circular velocity at the solar distance from the Galactic Centre is 220 $\mathrm{km\,s^{-1}}$ (i.e. at $R_0 = 8.0$ kpc from the centre).

When people first starting measuring rotation curves (c. 1970), it quickly became clear that the mass in disc galaxies does not follow the visible mass. It was found that disc galaxies generically have rotation curves that are fairly flat to as far out as they could be measured (out to several scale lengths). This is very different to the behaviour that would be expected were the visible mass – the mass of the stars and gas – the only matter in the galaxies. This is interpreted as strong evidence for the existence of dark matter in galaxies.

The simplest interpretation of a flat rotation curve is that based on the assumption the dark matter is spheroidally distributed in a 'dark halo'. For a spherical distribution of mass, $v_{circ} =$constant implies that the enclosed mass $M(r) \propto r$, and so $\rho(r) \propto 1/r^2$.

Rotation curves determined from optical spectra are generally limited to $\simeq$ few scale lengths (assuming an exponential density profile). These do provide important evidence of flat rotation curves. However, 21 cm radio observations can be followed out to significantly greater distances from the centres of spiral galaxies, using the emission from the atomic hydrogen gas. These HI observations provide powerful evidence of a constant circular velocity with radius, out to radial distances where the density of stars has declined to very low levels, providing strong evidence for the existence of extensive dark matter haloes.

As yet it is not clear exactly how far dark matter haloes extend. Neither is there a good estimate of the total mass of any disc galaxy. This is what makes disc rotation curves very important.

5.3 Theoretical Interpretation

However, one needs to be careful about interpreting flat rotation curves. The existence of dark matter haloes is a very important subject and caution is appropriate before accepting evidence that has profound significance to our understanding of matter in the Universe. For this reason, attempts were made to model observed rotation curves using as little mass in the dark matter haloes as possible. These 'maximal disc models' attempted to fit the observed data by assuming that the stars in the galactic discs had

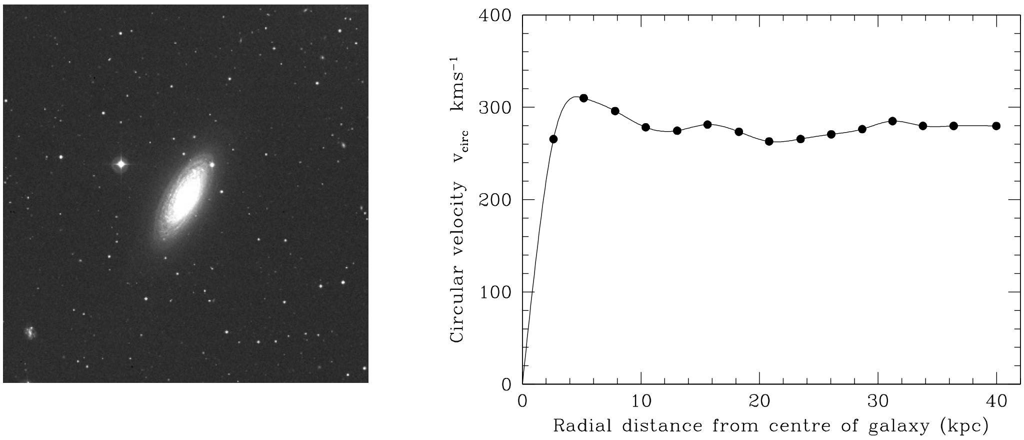

Figure 5.1: The spiral galaxy NGC 2841 and its H I 21 cm radio rotation curve. The figure on the left presents an optical (blue light) image of the galaxy, while that on the right gives the rotation curve in the form of the circular velocity plotted against radial distance. The optical image covers the same area of the galaxy as the radio observations: the 21 cm radio emission from the atomic hydrogen gas is detected over a much larger area than the galaxy covers in the optical image. [The optical image was created using Digitized Sky Survey II blue data from the Palomar Observatory Sky Survey. The rotation curve was plotted using data by A. Bosma (Astron. J., 86, 1791, 1981) taken from S. M. Kent (AJ, 93, 816, 1987).]

as much mass as could still be consistent with our understanding of stellar populations. They still, however, required a contribution from a dark matter halo at large radii when H I observations were taken into account.

Importantly, the maximum contribution to the rotation curve from an e^{-R/R_0} disc is not (as we might naively expect) at around R_0 from the centre but at around $2.5R_0$. Adding the effect of a bulge can easily give a fairly flat rotation curve to $4R_0$ without a dark halo. To be confident about the dark halo, one needs to have the rotation curve for $\gtrsim 5R_0$. In practice, that means H I measurements: optical rotation curves do not go out far enough to say anything conclusive about dark haloes.

The rest of this section is a more detailed working out of the previous paragraph. It follows an elegant derivation and explanation due to A. J. Kalnajs.

Consider an axisymmetric disc galaxy. Consider the rotation curve produced by the disc matter only (at this stage we shall not consider the contribution from the bulge or from the dark matter halo). This analysis will use a cylindrical coordinate system (R, ϕ, z) with $R = 0$ at the centre of the galaxy, and the disc centred around $z = 0$. Let the surface mass density of the disc be $\Sigma(R)$.

The gravitational potential in the plane of the disc at the point $(R, \phi, 0)$ is

$$\Phi(R) \quad = \quad -\, G \int_0^{\infty} R' \, \Sigma(R') \, \mathrm{d}R' \int_0^{2\pi} \frac{\mathrm{d}\phi}{\sqrt{R^2 + R'^2 - 2RR' \cos\phi}} \quad , \qquad (5.3)$$

found by integrating the contribution from volume elements over the whole disc. To make this tractable, let us first define a function $L(u)$ so that

$$L(u) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi}{\sqrt{1+u^2-2u\cos\phi}} \quad . \tag{5.4}$$

(The function within the integral can be expanded into terms called Laplace coefficients, which are explained in many old celestial mechanics books.)
This can be expanded as

$$L(u) = 1 + \frac{u^2}{4} + \frac{9}{64}u^4 + \frac{25}{256}u^6 + \frac{1225}{16384}u^8 + \mathrm{O}(u^{10}) \quad \text{for } u<1 \ , \tag{5.5}$$

either by expressing it as Laplace coefficients (which uses Legendre polynomials) or using a binomial expansion of the function in u, and then integrating each term in the expansion. The integration over ϕ in Equation 5.3 can be expressed in terms of $L(u)$ as

$$\begin{aligned} \int_0^{2\pi} \frac{\mathrm{d}\phi}{\sqrt{R^2+R'^2-2RR'\cos\phi}} &= \frac{2\pi}{R}\, L\big(\tfrac{R'}{R}\big) \quad \text{for } R'<R \\ &= \frac{2\pi}{R'}\, L\big(\tfrac{R}{R'}\big) \quad \text{for } R'>R \ , \end{aligned} \tag{5.6}$$

because the expansion of $L(u)$ assumed that $u<1$.
Splitting the integration in Equation 5.3 into two parts (for $R'=0$ to R, and for $R'=R$ to ∞) and substituting for $L(R'/R)$ and $L(R/R')$, we obtain,

$$\Phi(R) = -2\pi G \int_0^R \frac{R'}{R}\, \Sigma(R')\, L\big(\tfrac{R'}{R}\big)\, \mathrm{d}R' \; - 2\pi G \int_R^\infty \Sigma(R')\, L\big(\tfrac{R}{R'}\big)\, \mathrm{d}R' \ . \tag{5.7}$$

Consider a star in a circular orbit in the disc at radius R, having a velocity v. The radial component of the acceleration is

$$\frac{v^2}{R} = \frac{\partial\Phi}{\partial R} \ ,$$

and hence

$$\begin{aligned} v^2(R) &= R\,\frac{\partial\Phi}{\partial R} \\ &= -2\pi G R \frac{\mathrm{d}}{\mathrm{d}R} \int_0^R \frac{R'}{R}\, \Sigma(R')\, L\big(\tfrac{R'}{R}\big)\, \mathrm{d}R' \; - 2\pi G R \frac{\mathrm{d}}{\mathrm{d}R} \int_R^\infty \Sigma(R')\, L\big(\tfrac{R}{R'}\big)\, \mathrm{d}R' \end{aligned}$$

on substituting for Φ from Equation 5.7. These two differentials of integrals can be simplified by using a result known as Leibniz's Integral Rule, or Leibniz's Theorem for the differential of an integral. This states for a function f of two variables,

$$\frac{\mathrm{d}}{\mathrm{d}c} \int_{a(c)}^{b(c)} f(x,c)\, \mathrm{d}x = \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} f(x,c)\, \mathrm{d}x + f(b,c)\frac{\mathrm{d}b}{\mathrm{d}c} - f(a,c)\frac{\mathrm{d}a}{\mathrm{d}c} \ . \tag{5.8}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}R} \int_0^R \frac{R'}{R}\, \Sigma(R')\, L\big(\tfrac{R'}{R}\big)\, \mathrm{d}R' = \int_0^R \frac{\partial}{\partial R}\left[\frac{R'}{R} L\big(\tfrac{R'}{R}\big)\right] \Sigma(R')\, \mathrm{d}R' + \Sigma(R) L(1)$$

and $$\frac{\mathrm{d}}{\mathrm{d}R} \int_0^R \Sigma(R')\, L\big(\tfrac{R}{R'}\big)\, \mathrm{d}R' = \int_R^\infty \frac{\partial}{\partial R}\left[L\big(\tfrac{R}{R'}\big)\right] \Sigma(R')\, \mathrm{d}R' - \Sigma(R) L(1) \ .$$

Therefore we get

$$v^2(R) \;=\; -\,2\pi G R \int_0^R \frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R'}{R}\,L\big(\tfrac{R'}{R}\big)\right)\Sigma(R')\;\mathrm{d}R' \;-\; 2\pi G R\int_R^\infty \Sigma(R')\,\frac{\mathrm{d}}{\mathrm{d}R}L\big(\tfrac{R}{R'}\big)\,\mathrm{d}R'$$

But,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{R'}{R}\,L\big(\tfrac{R'}{R}\big)\right) &= -\,\frac{R'}{R^2}\,L\big(\tfrac{R'}{R}\big) \;+\; \frac{R'}{R}\,\frac{\mathrm{d}}{\mathrm{d}R}L\big(\tfrac{R'}{R}\big) && \text{from the product rule}\\
&= -\,\frac{R'}{R^2}\,L\big(\tfrac{R'}{R}\big) \;+\; \frac{R'}{R}\,\frac{\mathrm{d}L\big(\tfrac{R'}{R}\big)}{\mathrm{d}(R'/R)}\,\frac{\mathrm{d}(R'/R)}{\mathrm{d}R} && \text{from the chain rule}\\
&= -\,\frac{R'}{R^2}\,L\big(\tfrac{R'}{R}\big) \;-\; \frac{R'^2}{R^3}\,L'\big(\tfrac{R'}{R}\big) && \text{writing } L'(u)\equiv\tfrac{\mathrm{d}L(u)}{\mathrm{d}u}\ .
\end{aligned}$$

$$\begin{aligned}
\therefore\quad v^2 \;=\; &+\,2\pi G\int_0^R\left[\frac{R'}{R}L\big(\tfrac{R'}{R}\big)+\big(\tfrac{R'}{R}\big)^2L'\big(\tfrac{R'}{R}\big)\right]\;\Sigma(R')\,\mathrm{d}R'\\
&-\,2\pi G\int_R^\infty\big(\tfrac{R}{R'}\big)\;L'\big(\tfrac{R}{R'}\big)\;\Sigma(R')\,\mathrm{d}R'\ .
\end{aligned} \tag{5.9}$$

This can be quite messy and it can abbreviated as

$$\boxed{v^2(R) \;=\; 2\pi G\int_0^\infty K\big(\tfrac{R}{R'}\big)\;\Sigma(R')\;\mathrm{d}R'\ ,} \tag{5.10}$$

where the function $K\big(\frac{R}{R'}\big)$ represents the function over both $R'=0$ to R and $R'=R$ to ∞ domains.
Changing variables to $x\equiv\ln R, y\equiv\ln R'$, we can write this as a convolution

$$v^2(R) \;=\; 2\pi G\int_{-\infty}^{\infty} K(e^{x-y})\,R'\,\Sigma(R')\,\mathrm{d}y\ . \tag{5.11}$$

The kernel $K(R/R')$ is in Figure 5.2.

Figure 5.3 shows $R\,\Sigma(R)$ and v^2 for an exponential disc, but the general shapes are not very sensitive to whether $\Sigma(R)$ is precisely exponential. The important qualitative fact is that whatever $R\,\Sigma(R)$ does, v^2 does roughly the same, but expanded by a factor of $\simeq e$.

The distinctive shape of the $v^2\,(\ln R)$ curve for realistic discs makes it very easy to recognise *non*-disc mass. Figure 5.4, following Kalnajs, shows the rotation curves you get by adding either a bulge or a dark halo. (Actually this figure fakes the bulge/halo contribution by adding a smaller/larger disc; but if you properly add spherical mass distributions for disc/halo, the result is very similar.) Kalnajs's point is that a bulge+disc rotation curve has a similar shape to a disc+halo rotation curve – only the scale is different. So when examining a flat(-ish) rotation curve, you must ask what the disc scale radius is.

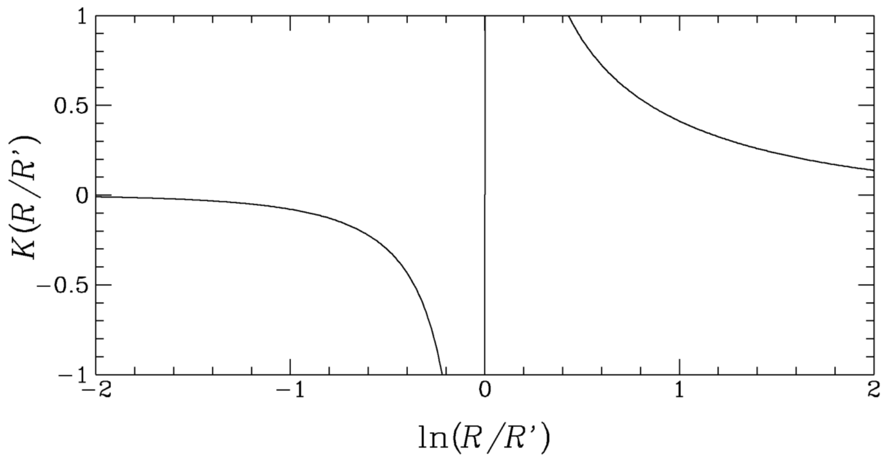

Figure 5.2: The kernel $K(R/R')$. Observe that the $R > R'$ part tends to have higher absolute value than the $R < R'$ part.

5.4 Representing Dark Matter Distributions

The dark matter within spiral galaxies does not appear to be confined to the discs and it is probably distributed approximately spheroidally. A popular density profile that has been adopted for modelling dark matter haloes has the form

$$\rho(r) \;=\; \frac{\rho_0}{1 \,+\, (r/a)^2} \;, \tag{5.12}$$

where r is the radial distance from the centre of the galaxy, ρ_0 is the central dark matter density, and a is a constant. This form does reproduce the observed rotation curves of spiral galaxies adequately: it gives a circular velocity that is $v_{circ} = 0$ at $R = 0$, that rises rapidly with the radial distance in the plane of the disc R, and then becomes flat (v_{circ} =constant) for $R \gg a$. This profile, however, has the problem that its mass is infinite. Therefore a more practical functional form might seem to be

$$\rho(r) \;=\; \frac{\rho_0}{1 \,+\, (r/a)^n} \;, \tag{5.13}$$

where a and n are constants, with $n > 3$ giving a finite mass. However, $n > 3$ would no longer give flat rotation curves. Therefore a better option might be to change Equation 5.12 by incorporating an extra component that truncates it at very large radii.

Some numerical N-body simulations of galaxy formation have predicted that dark matter haloes will have density profiles of the form

$$\rho(r) \;=\; \frac{k}{r\,(a \,+\, r)^2} \;, \tag{5.14}$$

where a and k are constants. This is known as the Navarro-Frenk-White profile after the scientists who first described it. It fits the densities of collections of particles representing dark matter haloes in numerical simulations, and does so adequately over broad ranges in masses and sizes. It is therefore often used to represent the dark matter haloes of galaxies and also of clusters of galaxies.

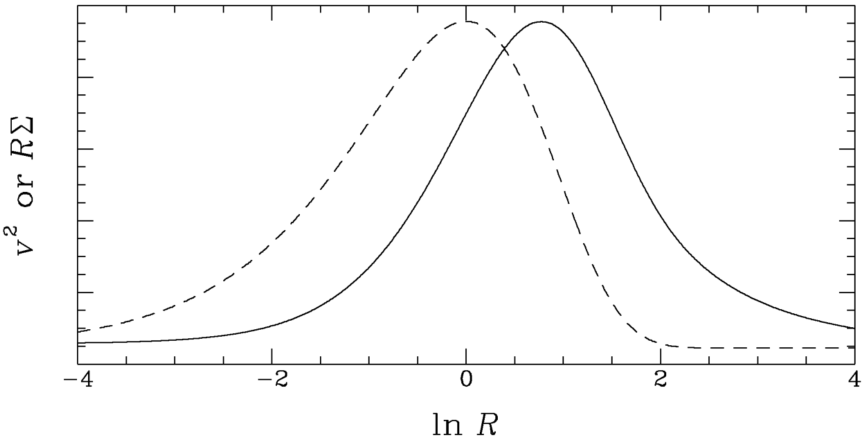

Figure 5.3: The effect of an exponential disc. The dashed curve is $R\,\Sigma(R)$ for an exponential disc with $\Sigma \propto e^{-R}$ and the solid curve is $v^2(R)$. Note that R is measured in disc scale lengths, but the vertical scales are arbitrary.

The profiles above are spherical: the density depends only on the radial distance r from the centre. These functional forms for ρ can be modified to allow for flattened systems.

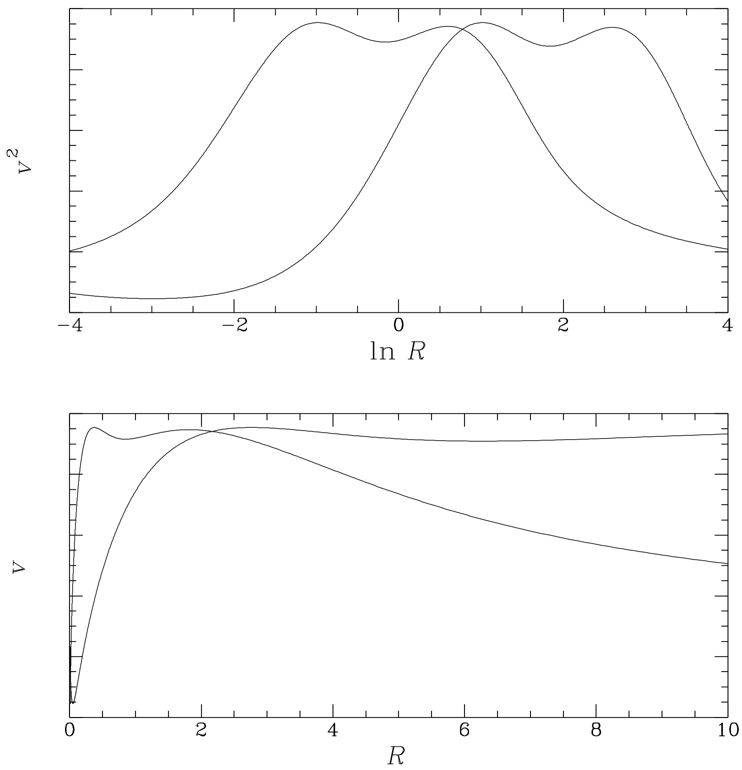

Figure 5.4: Plots of v^2 against $\ln R$ (upper panel) or v against R (lower panel). For one curve in each panel, a second exponential disc with mass and scale radius both scaled down by $e^2 \simeq 7.39$ has been added (to mimic a bulge); for the other curve a second exponential disc with mass and scale radius both scaled up by $e^2 \simeq 7.39$ has been added (to mimic a dark halo).