The Third Jeans Equation

To derive the third Jeans Equations, we firstly introduce a tensor velocity dispersion σ_{ij} defined so that

$$n\,\sigma_{ij}^2 \;\equiv\; \int \big(v_i - \langle v_i\rangle\big)\,\big(v_j - \langle v_j\rangle\big)\; f\;\mathrm{d}^3\mathbf{v}$$

$$\textbf{for } i,j = 1,3$$

σ_{ij} is a symmetric tensor representing the spread of velocities in each direction

We can always choose some coordinate system in which σ_{ij} is diagonal i.e. $\sigma_{11} \neq 0,\ \sigma_{22} \neq 0,\ \sigma_{33} \neq 0$, but all the other elements are zero

This is known as the *velocity ellipsoid*

Choice of components according to the coordinate system e.g.:

σ_{xx}, σ_{yy} **and** σ_{zz} **for** (x, y, z)
σ_{RR}, $\sigma_{\phi\phi}$ **and** σ_{zz} **for** (R, ϕ, z)
σ_{rr}, $\sigma_{\theta\theta}$ **and** $\sigma_{\phi\phi}$ **for** (r, θ, ϕ)

If the velocity dispersion is isotropic:

$\sigma_{11} = \sigma_{22} = \sigma_{33}$ **and** $\sigma_{ij} = 0$ **for** $i \neq j$

which we might simplify by writing as σ only

Rearranging $n\,\sigma_{ij}^2 \;\equiv\; \int \big(v_i - \langle v_i\rangle\big)\,\big(v_j - \langle v_j\rangle\big)\; f\;\mathrm{d}^3\mathbf{v}$ and multiplying out:

$$
\begin{aligned}
\sigma_{ij}^2 \;&=\; \frac{1}{n}\int \big(v_i - \langle v_i\rangle\big)\,\big(v_j - \langle v_j\rangle\big)\, f\;\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n}\int \Big(\; v_i v_j - v_i\langle v_j\rangle - \langle v_i\rangle v_j + \langle v_i\rangle\,\langle v_j\rangle \;\Big)\; f\;\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n}\int v_i v_j\; f\;\mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n}\int v_i\,\langle v_j\rangle\; f\;\mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n}\int \langle v_i\rangle\, v_j\; f\;\mathrm{d}^3\mathbf{v} \\
&\qquad\qquad + \frac{1}{n}\int \langle v_i\rangle\,\langle v_j\rangle\; f\;\mathrm{d}^3\mathbf{v} \\
&=\; \frac{1}{n}\int v_i v_j\; f\;\mathrm{d}^3\mathbf{v} \;-\; \langle v_j\rangle\,\frac{1}{n}\int v_i\; f\;\mathrm{d}^3\mathbf{v} - \;\langle v_i\rangle\,\frac{1}{n}\int v_j\; f\;\mathrm{d}^3\mathbf{v} \\
&\qquad\qquad + \langle v_i\rangle\,\langle v_j\rangle \frac{1}{n}\int f\;\mathrm{d}^3\mathbf{v}
\end{aligned}
$$

(because $\langle v_i\rangle$, $\langle v_j\rangle$ are functions of x, t only)

$$
=\; \langle v_i v_j\rangle \;-\; \langle v_j\rangle\,\langle v_i\rangle \;-\; \langle v_i\rangle\,\langle v_j\rangle \;+\; \langle v_i\rangle\,\langle v_j\rangle
$$

where in the last step, we have used:

$$
\begin{aligned}
n \;&\equiv\; \int f\;\mathrm{d}^3\mathbf{v} \\
n\,\langle v_i\rangle \;&\equiv\; \int v_i\, f\;\mathrm{d}^3\mathbf{v}
\end{aligned}
$$

$$
\therefore \qquad \boxed{\;\sigma_{ij}^2 \;=\; \langle v_i v_j\rangle \;-\; \langle v_i\rangle\,\langle v_j\rangle\;}
$$

This can be rearranged in terms of $\langle v_i v_j\rangle$:

$$
\langle v_i v_j\rangle \;=\; \sigma_{ij}^2 \;+\; \langle v_i\rangle\,\langle v_j\rangle
$$

The second Jeans equation is given by:

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\langle v_i v_j\rangle\big) \;=\; -\frac{\partial\Phi}{\partial x_i}\,n$$

for each of $i = 1, 2, 3$

Substituting for $\langle v_i v_j\rangle$**, from the previous expression, we get:**

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} \;+\; \sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big) \;+\; \frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big)\right] \;=\; -\frac{\partial\Phi}{\partial x_i}n$$

for each of $i = 1, 2, 3$

The first term on the LHS can be expanded using the product rule:

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} \;=\; \langle v_i\rangle\frac{\partial n}{\partial t} + n\,\frac{\partial\langle v_i\rangle}{\partial t}$$

Substituting this and writing the sums separately, we get :

$$\langle v_i\rangle\frac{\partial n}{\partial t} \;+\; n\,\frac{\partial\langle v_i\rangle}{\partial t} + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big) \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big) \;=\; -\frac{\partial\Phi}{\partial x_i}\,n \qquad (6)$$

We can eliminate the 1st and 4th terms by going back to the the first Jeans Equation, which is:

$$\frac{\partial n}{\partial t} \;+\; \sum_{i=1}^{3}\frac{\partial n\langle v_i\rangle}{\partial x_i} \;=\; 0$$

Multiplying the first Jeans equation through by $\langle v_i \rangle$:

$$\langle v_i \rangle \frac{\partial n}{\partial t} \;+\; \langle v_i \rangle \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\big(n\langle v_j \rangle\big) \;=\; 0$$

$$\therefore \quad \langle v_i \rangle \frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3} \langle v_i \rangle \frac{\partial}{\partial x_j}\big(n\langle v_j \rangle\big) \;=\; 0$$

Also, using the product rule, we note that :

$$\frac{\partial}{\partial x_j}\big(n\langle v_i \rangle\langle v_j \rangle\big) \;=\; \langle v_i \rangle \frac{\partial}{\partial x_j}\big(n\langle v_j \rangle\big) \;+\; n\,\langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j}$$

Or, rearranging :

$$\langle v_i \rangle \frac{\partial}{\partial x_j}\big(n\langle v_j \rangle\big) \;=\; \frac{\partial}{\partial x_j}\big(n\langle v_i \rangle\langle v_j \rangle\big) \;-\; n\,\langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j}$$

Substituting this back in to the first Jeans equation (top) gives:

$$\langle v_i \rangle \frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j}\big(n\langle v_i \rangle\langle v_j \rangle\big) \;-\; n\,\langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j} \right) \;=\; 0$$

$$\therefore \quad \langle v_i \rangle \frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\big(n\langle v_i \rangle\langle v_j \rangle\big) \;=\; n \sum_{j=1}^{3} \langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j}$$

Finally, substituting this back into equation (6) :

$$\boxed{\,n \frac{\partial \langle v_i \rangle}{\partial t} \;+\; n \sum_{j=1}^{3} \langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j} \;=\; -\,n \frac{\partial \Phi}{\partial x_i} \;-\; \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big)\,}$$

for each of $i = 1, 2$ or 3

This is the third Jeans Equation

The third Jeans equation can also be expressed as:

$$\frac{\mathrm{d}\langle \mathbf{v} \rangle}{\mathrm{d}t} \;=\; -\,\boldsymbol{\nabla}\Phi \;-\; \frac{1}{n}\,\boldsymbol{\nabla}\boldsymbol{\cdot}(n\boldsymbol{\sigma}^2)$$

where

$\langle \mathbf{v} \rangle$ **is the mean velocity vector**
t **is the time**
Φ **is the potential**
n **is the number density of stars**
$\boldsymbol{\sigma}^2$ **represents the tensor** σ^2_{ij}

Note that here $\mathrm{d}/\mathrm{d}t$ **is not** $\partial/\partial t$**, but**

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \;\left(\equiv \frac{D\mathbf{v}}{Dt}\right) \;=\; \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\mathbf{v}$$

which is sometimes called the convective derivative – it is also sometimes written as D/Dt **to emphasise that it is not simply** $\frac{\partial}{\partial t}$

The third Jeans equation is similar to the Euler equation in fluid dynamics

For an ordinary fluid:

$$\frac{\mathrm{d}\langle \mathbf{v} \rangle}{\mathrm{d}t} \;=\; -\,\boldsymbol{\nabla}\Phi \;-\; \frac{\boldsymbol{\nabla}P}{\rho} \;+\; \textbf{viscous terms}$$

Pressure P **arises because of the high rate of molecular encounters (and** P **is isotropic - a scalar)**

In stellar dynamics, the stars behave like a fluid in which $\rho\boldsymbol{\sigma}^2$ **behaves like a pressure, except it is anisotropic (a tensor)**

The Jeans Equations in cartesian coordinates : summary

In cartesian coordinates (x_1, x_2, x_3)

The first Jeans Equation is:

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial n \langle v_i \rangle}{\partial x_i} = 0$$

The second Jeans Equation is:

$$\frac{\partial (n \langle v_i \rangle)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \langle v_i v_j \rangle \big) = - \frac{\partial \Phi}{\partial x_i} n$$

for each of $i = 1, 2, 3$

The third Jeans Equation is:

$$n \frac{\partial \langle v_i \rangle}{\partial t} + n \sum_{j=1}^{3} \langle v_j \rangle \frac{\partial \langle v_i \rangle}{\partial x_j} = - n \frac{\partial \Phi}{\partial x_i} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \sigma_{ij}^2 \big)$$

where i **can be any of 1, 2 or 3 and**

$$\sigma_{ij}^2 = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

The Jeans Equations in a spherically symmetric system

For a spherically-symmetric galaxy:

$$\partial/\partial\phi = 0 \text{ and } \partial/\partial\theta = 0$$

If it is also in a steady state:

$$\partial/\partial t = 0$$

The *second* Jeans Equation in spherical polar coordinate system (r, θ, ϕ) **then simplifies to:**

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n \, \langle v_r^2 \rangle \Big) \; + \; \frac{n}{r} \left[\, 2\langle v_r^2 \rangle \; - \; \langle v_\theta^2 \rangle \; - \; \langle v_\phi^2 \rangle \, \right] \; = \; - \, n \, \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

This might be used, for example, for a spherical elliptical galaxy or the halo of our Galaxy

We can calculate the gradient, $\mathrm{d}\Phi/\mathrm{d}r$, **in the potential in this spherical case very simply:**

acceleration due to gravity is $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ **(always)**

$g = -GM(r)/r^2$ **in a spherically symmetric system where** $M(r)$ **is the mass interior to the radius** r

and $\boldsymbol{\nabla}\Phi = \mathrm{d}\Phi/\mathrm{d}r$ **in a spherical system**

Giving $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r^2$

For an isotropic velocity distribution:

$$\langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = \sigma^2(r)$$

where σ is the velocity dispersion

In the special case of an isotropic velocity distribution in a spherically symmetric potential, the second Jeans Equation in spherical polar coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(n \, \langle v_r^2 \rangle \Big) \; + \; \frac{n}{r} \left[\, 2\langle v_r^2 \rangle \; - \; \langle v_\theta^2 \rangle \; - \; \langle v_\phi^2 \rangle \, \right] \; = \; - \, n \, \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

then reduces to the following:

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(n \, \sigma^2 \right) \; + \; \frac{n}{r} \, (0) \; = \; - \, n \, \frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

or

$$\boxed{\sigma^2 \, \frac{\mathrm{d}n}{\mathrm{d}r} \; = \; - \, n \, \frac{\mathrm{d}\Phi}{\mathrm{d}r}}$$

The Jeans Equations in an axisymmetric System

Useful for disc galaxies

Assuming axisymmetry (so $\partial/\partial\phi = 0$), and a steady state (so $\partial/\partial t = 0$), the first Jeans Equation in cylindrical coordinates (R, ϕ, z) is:

$$\frac{\partial n}{\partial t} \;+\; \frac{1}{R}\frac{\partial}{\partial R}\big(Rn\langle v_R\rangle \big) \;+\; \frac{\partial}{\partial z}\big(n\langle v_z\rangle \big) \;=\; 0$$

The second Jeans Equation in cylindrical coordinates for an axisymmetric system is:

$$\frac{\partial}{\partial t}(n\langle v_R\rangle) + \frac{\partial}{\partial R}(n\langle v_R^2\rangle) + \frac{\partial}{\partial z}(n\langle v_R v_z\rangle) + \frac{n}{R}\Big(\langle v_R^2\rangle - \langle v_\phi^2\rangle\Big) = -\,n\,\frac{\partial \Phi}{\partial R}$$

for the R direction

$$\frac{\partial}{\partial t}(n\langle v_\phi\rangle) + \frac{\partial}{\partial R}(n\langle v_R v_\phi\rangle) + \frac{\partial}{\partial z}(n\langle v_\phi v_z\rangle) + \frac{2\,n}{R}\langle v_R v_\phi\rangle = 0 \qquad (7)$$

for the ϕ direction

$$\frac{\partial}{\partial t}\big(n\langle v_z\rangle \big) + \frac{\partial}{\partial R}\big(n\langle v_R v_z\rangle \big) + \frac{\partial}{\partial z}\big(n\langle v_z^2\rangle \big) + \frac{n\,\langle v_R v_z\rangle}{R} = -\,n\,\frac{\partial \Phi}{\partial z}$$

for the z direction

Using the Jeans Equations

The Jeans Equations have been represented here in terms of the number density $n(\mathbf{x}, t)$ of stars

However, it is possible to work instead with the mean mass density in space, ρ, instead of n

The Jeans Equations can be used for ALL stars in a galaxy

But sometimes they are used for subpopulations in our Galaxy (e.g. G dwarfs, K giants)

If they are used for subpopulations, Φ remains the total gravitational potential of all matter (including dark matter), but the velocities and number densities refer to the subpopulations

Subpopulations may contribute little to the overall density - trace populations

Example of the use of the Jeans Equations: the surface mass density of the Galactic disc

We can use the Jeans equations to estimate the surface mass density Σ of the Galactic disc at the solar distance from the Galactic centre

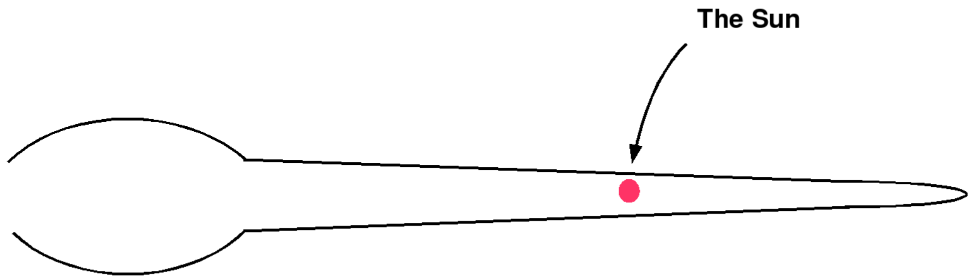
Surface mass density is mass per unit area of the disc expressed in units of $\mathrm{kg\,m^{-2}}$, or more commonly $M_\odot\,\mathrm{pc}^{-2}$

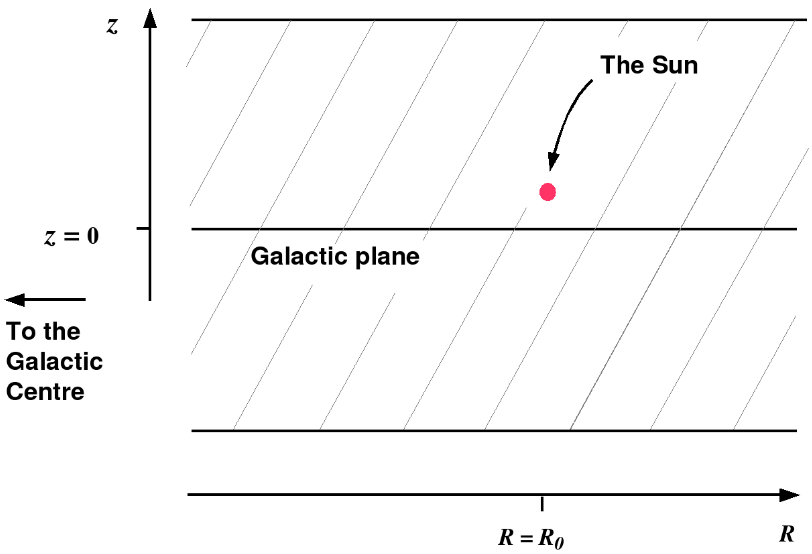
$\Sigma = \Sigma(R, \phi) = \Sigma(R)$ for axisymmetry

Use observations of velocities of stars along line of sight lying some distance above or below the Galactic plane

Allows estimate of quantity of dark matter in the disc – important constraint on nature of dark matter

Need to calculate Σ from the gravitational potential Φ to account for ALL the mass (stars/gas AND dark matter) – integrating n would only give the mass of the visible stars





The second Jeans Equation in cylindrical coordinates (R, ϕ, z) centred on the Galaxy, with $z = 0$ in the plane and $R = 0$ at the Galactic Centre states for the z direction that:

$$\frac{\partial(\, n\langle v_z\rangle\,)}{\partial t} + \frac{\partial(\, n\langle v_R v_z\rangle\,)}{\partial R} + \frac{\partial(\, n\langle v_z^2\rangle\,)}{\partial z} + \frac{n\langle v_R v_z\rangle}{R} = -n\frac{\partial \Phi}{\partial z}$$

where n is the star number density
v_R and v_z are the velocity components in the R and z directions
$\Phi(R, z, t)$ is the Galactic gravitational potential
t is time.

Apply observational and practical constraints to simplify the problem:

– assume steady state → n independent of time t, so the first term $\partial(n\langle v_z\rangle)/\partial t = 0$

– observations show that:

$$\frac{\partial(\, n\langle v_R v_z\rangle\,)}{\partial R} \;\simeq\; 0 \quad \textbf{and} \quad \frac{n\langle v_R v_z\rangle}{R} \;\simeq\; 0$$

as is to be expected because of the cancelling of positive and negative terms of the z-components of the velocity, therefore

$$\frac{\partial(\, n\langle v_z^2\rangle\,)}{\partial z} \;=\; -\,n\,\frac{\partial\Phi}{\partial z}$$

where $\langle v_z^2\rangle$ is the mean square velocity in the direction perpendicular to the Galactic plane

Poisson's equation gives $\nabla^2\Phi = 4\pi G\rho$, where ρ is the total mass density at a point

In cylindrical coordinates the Laplacian is:

$$\nabla^2\Phi \;=\; \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi}{\partial R}\right) \;+\; \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} \;+\; \frac{\partial^2\Phi}{\partial z^2}$$

(from Lecture 1)

If we observe stars directly above and below the Galactic plane all at similar Galactocentric radius R and ϕ, we can neglect the $\partial\Phi/\partial R$ and $\partial^2\Phi/\partial\phi^2$ terms

$$\therefore \quad \frac{\partial^2\Phi}{\partial z^2} \;=\; 4\pi G\rho$$

and substituting for $\partial\Phi/\partial z$ from the Jeans equation:

$$\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right) \;=\; 4\pi G\rho$$

However we need Σ not ρ

Integrating perpendicular to the Galactic plane from $-z$ to z, the surface mass density within a distance z of the plane at a Galactocentric radius R is:

$$\begin{aligned}
\Sigma(R,z) &\equiv \int_{-z}^{z} \rho \, \mathrm{d}z' \\
&= \int_{-z}^{z} \frac{1}{4\pi G}\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right)\,\mathrm{d}z' \\
&= -\frac{1}{4\pi G}\left[\frac{1}{n}\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\right]_{z'=-z}^{z} \\
&= -\frac{1}{2\pi G n}\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\bigg|_{z}
\end{aligned}$$

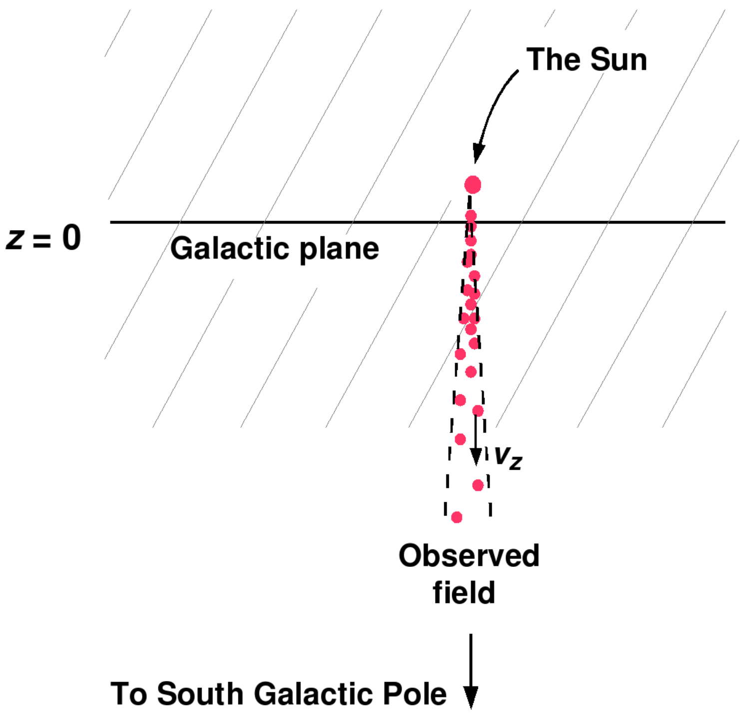
assuming symmetry about $z = 0$

So the surface mass density within a distance $\pm z$ of the plane at the solar Galactocentric radius R_0 is :

$$\Sigma(R_0,z) \;=\; -\frac{1}{2\pi G n}\,\frac{\partial}{\partial z}\Big(n\langle v_z^2\rangle\Big)\bigg|_{z}$$

If the star densities n can be measured as a function of height z from the plane and if the z-component of the velocities v_z can be measured as spectroscopic radial velocities, we can solve for $\Sigma(R_0, z)$ as a function of z

This gives, after modelling the contribution from the dark matter halo, the surface mass density of the Galactic disc



The analysis can be performed on some subclass of stars, such as G giants or K giants. In this case the number density n of stars in space is that of the subclass. Number counts of stars towards the Galactic poles, combined with estimates of the distances to individual stars, give n

Spectroscopic observations give radial velocities (the velocity components along the line of sight) through the Doppler effect. By observing towards the Galactic poles, the radial velocities are the same as the v_z components

This analysis gives $\Sigma(R_0, z)$ as a function of z

The value increases with z as a greater proportion of the stars of the disc are included, until all the disc matter is included

$\Sigma(R_0, z)$ will still increase slowly with z beyond this as an increasing amount mass from the dark matter halo is included - need to determine the contribution $\Sigma_d(R_0)$ from the disc alone to the observed data

Additional complication is that in measuring $\partial(n\langle v_z^2\rangle)/\partial z$ as a function of z, we are dealing with the differential of observed quantities - so the effects of observational errors can be considerable

First done by Oort (1932)

Repeated in the 1980s by Bahcall and by Kuijken and Gilmore (1989, MNRAS, 239, 605):

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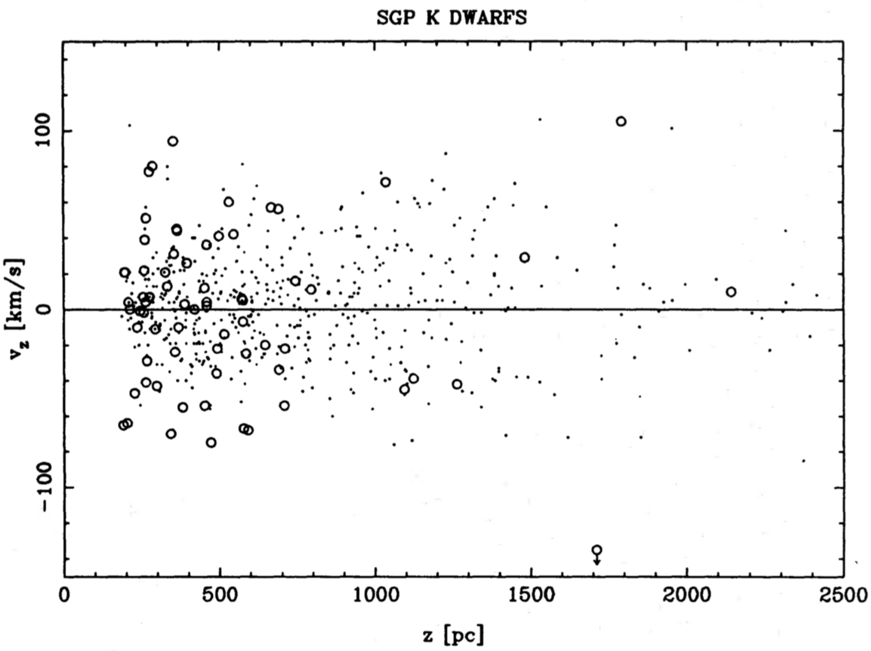

Figure 15. The velocity distributions of dwarfs (dots) and giants (circles), plotted against the dwarf photometric parallax. Note that the giant stars are predominantly bright, and hence appear close to the plane, and that their velocity dispersion is greater than that of the dwarfs near the plane.

Considerable debate about the interpretation of results

Early studies claimed evidence of dark matter in Galactic disc

But more recently some consensus has developed that there is little dark matter in the disc itself (apart from contribution from the dark matter halo that extends into the disc)

A modern value is:

$$\Sigma_d(R_0) = 50 \pm 10\ M_\odot\ \mathrm{pc}^{-2}$$

Absence of significant dark matter in disc indicates that dark matter does not follow baryonic matter closely on a small scale - a very important result