The Jeans Theorem

The Jeans Theorem : an important result in stellar dynamics that states the importance of integrals of the motion in solving the CBE for gravitational potentials that do not change with time

Named after its discoverer, the English astronomer, physicist and mathematician Sir James Hopwood Jeans (1877–1946)

It states that:

Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of the motion in the galaxy's potential, and any function of the integrals yields a steady-state solution of the CBE

This implies that in a potential that does not change with time, we can express the CBE in terms of integrals of motion, and then solve for the distribution function f in terms of those integrals of motion

We can then convert the solution of f in terms of the integrals to a solution for f in terms of the space and velocity coordinates

For example:

If the energy per unit mass E_m and total angular momentum components L_x and L_y are constant for each star in some potential

Then we can solve for f uniquely as a function of E_m, L_x and L_y

We can then convert from E_m, L_x and L_y to give f as a function of (x, y, z, v_x, v_y, v_z)

Solving for *f* in spherical galaxies

The Jeans Theorem does apply in spherical systems of stars, such as spherical elliptical galaxies

So in a spherical system, f can depend on (at most) three integrals of motion

Simplest case is for f to be a function of the energy of the stars only

(Since we are considering bound systems, then $f = 0$ for $E > 0$ i.e. any stars that did have $E > 0$ will have escaped from the galaxy)

To find an equilibrium solution, we only have to satisfy Poisson's equation $\nabla^2\Phi = 4\pi G\rho$

The total energy of a star of mass m moving with a velocity v is

$$E = \frac{1}{2}mv^2 + m\Phi$$

where Φ is the gravitational potential at the stars position

More convenient to use the *energy per unit mass* :

$$E_m = \frac{1}{2}v^2 + \Phi$$

Also, given spherical symmetry, it's easier to use spherical polar coordinates (r, θ, ϕ), with the origin at the centre

Poisson's equation relates the Laplacian of the gravitational potential Φ at a point to the local mass density ρ as $\nabla^2\Phi = 4\pi G\rho$

In spherical polar coordinates, the Laplacian of any scalar function $A(r, \theta, \phi)$ is

$$\nabla^2 A \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2}$$

(a standard result from vector calculus: see Lecture 1)

In a spherically symmetric galaxy that does not change with time, the potential is a function of the radial distance r from the centre only

So $\partial\Phi/\partial\theta = 0$ and $\partial\Phi/\partial\phi = 0$, and therefore

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$$

Substituting this into the Poisson equation gives

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho$$

The distribution function f is related to the number density n of stars by

$$n = \int f \, \mathrm{d}^3\mathbf{v}$$

and in this case f is a function of energy per unit mass: $f = f(E_m)$

We can relate this to the density ρ using $\rho = \overline{m}\, n$ where $\overline{m}$ is the mean mass of a star, giving

$$\rho = \overline{m} \int f \, \mathrm{d}^3\mathbf{v}$$

(note that here we are assuming that mass is in the form of stars only: no dark matter)

The integral is over all velocities at a particular position

Convert from d^{3}v to dv, where $v \equiv |\mathbf{v}|$ by considering a thin spherical shell in a space defined by the three velocity components, giving d^{3}v $= 4\pi v^2 \mathrm{d}v$

So

$$\rho = 4\pi \, \overline{m} \int f v^2 \, \mathrm{d}v \qquad (3)$$

What about the integration limits (range of v)?

For any particular point in the galaxy (any value of r), the minimum possible velocity is $v = 0$ (when a star moving on a radial orbit reaches its maximum distance from the centre at that point)

The maximum velocity at this position occurs when a star has the greatest possible energy, $E_m = 0$, which would allow a star to move out from the point to arbitrary distance

(any star with energy per unit mass $E_m > 0$ will be moving faster than the escape velocity for that location and will escape from the galaxy)

Since $E_m = \frac{1}{2}v^2 + \Phi(r)$, the maximum velocity (corresponding to $E_m = 0$) is $v = \sqrt{-2\Phi(r)}$

(note $\Phi(r)$ is negative, so $-2\Phi(r)$ is positive)

So we get:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \quad = \quad (4\pi)^2\,G\,\overline{m}\,\int_0^{\sqrt{-2\Phi(r)}} f\,v^2\,\mathrm{d}v \qquad (4)$$

Alternatively, convert this integral over velocity to an integral over energy per unit mass:

$E_m = \frac{1}{2}v^2 + \Phi$ gives d$E_m = v\,\mathrm{d}v$ at a fixed position (and hence for a constant $\Phi(r)$), also $v = \sqrt{2(E_m - \Phi)}$

So at any radius r

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \quad = \quad (4\pi)^2\sqrt{2}\,G\overline{m}\,\int_{\Phi(r)}^{0} \sqrt{E_m - \Phi(r)}\;f(E_m)\,\mathrm{d}E_m \qquad (5)$$

In equation (4) normally $f(v)$ is given – try to solve for $\Phi(r)$ and hence $\rho(r)$; a nonlinear differential equation

In equation (5) we would normally take Φ as given, and try to solve for $f(E_m)$; a linear integral equation

There are several $f(E_m)$ models in the literature

Can create a new one by picking some $\rho(r)$, computing $\Phi(r)$ and then solving equation (5) numerically

Examples for spherical, isotropic distribution functions:

(1) The Plummer Potential

As discussed earlier, the Plummer potential has a gravitational potential Φ and a mass density ρ at a radial distance r from the centre that are given by

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2+a^2}} , \qquad \rho(r) = \frac{3M_{tot}}{4\pi} \frac{a^2}{(r^2+a^2)^{5/2}}$$

where M_{tot} is the total mass of the galaxy and a is a constant

The distribution function f for the Plummer model is related to the density ρ by equation (3)

It can be shown that these $\Phi(r)$ and $\rho(r)$ forms give a solution

$$f(E_m) = \frac{24\sqrt{2}}{7\pi^3} \frac{a^2}{G^5 M_{tot}^4 \, \overline{m}} (-E_m)^{\frac{7}{2}}$$

This can be verified by inserting in equation (5)

This gives the distribution function f as a function only of the energy per unit mass E_m

To calculate f for any point (x, y, z, v_x, v_y, v_z) in phase space, we need only to calculate E_m from these coordinates and then calculate the value of f associated with that E_m

(2) The Isothermal Sphere

The isothermal sphere is defined by analogy with a Maxwell-Boltzmann gas

The distribution function as a function of the energy per unit mass E_m is

$$f(E_m) = \frac{n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{E_m}{\sigma^2}\right) = \frac{n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\frac{1}{2}v^2 + \Phi}{\sigma^2}\right)$$

where σ is a velocity dispersion and acts like temperature in a gas, and n_0 is a constant

Integrating over velocities

$$\begin{aligned} n(r) &= \int f \, \mathrm{d}^3\mathbf{v} = \int_0^\infty f \,.\, 4\pi \, v^2 \, \mathrm{d}v \\ &= \frac{4\pi \, n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\Phi}{\sigma^2}\right) \int_0^\infty v^2 \exp\left(-\frac{v^2}{2\sigma^2}\right) \mathrm{d}v \\ &= \frac{4\pi \, n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\Phi}{\sigma^2}\right) . \left(\frac{\sigma^3}{4}\sqrt{8\pi}\right) \\ &= n_0 \, \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) \end{aligned}$$

using the standard integral $\int_0^\infty x^2 \mathrm{e}^{-ax^2} \mathrm{d}x = \sqrt{\pi/a^3}/4$

Note the isothermal distribution includes stars with speeds from $v = 0$ to ∞, hence the integration limits

In practice, no stable galaxy will have stars with speeds larger than the escape velocity $\sqrt{-2\Phi(r)}$

Converting this to density $\rho(r)$ using $\rho = \overline{m}\, n$, where $\overline{m}$ is the mean mass of the stars, we get

$$\rho(r) \;=\; \rho_0 \, \exp\left(-\frac{\Phi(r)}{\sigma^2}\right)$$

and equivalently

$$\Phi(r) \;=\; -\,\sigma^2 \, \ln\left(\frac{\rho(r)}{\rho_0}\right)$$

where ρ_0 is a constant (with $\rho_0 \equiv n_0\,\overline{m}$)

Using this, Poisson's equation ($\nabla^2\Phi = 4\pi G\rho$) in a spherically symmetric potential on substituting for $\mathrm{d}\Phi/\mathrm{d}r$ becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\left(-\,\sigma^2 \ln\left(\frac{\rho}{\rho_0}\right)\right)\right) \;=\; 4\pi G\,\rho$$

which simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) \;=\; -\,\frac{4\pi G}{\sigma^2}\,r^2\,\rho$$

This is a second-order differential equation in ρ and r

One solution is

$$\rho(r) \;=\; \frac{\sigma^2}{2\pi G r^2}$$

known as the singular isothermal sphere

Although it has infinite mass, the isothermal sphere is often used as a model for dark matter haloes of disc galaxies, with some large-r truncation assumed

Observable and measurable quantities

f usually very difficult to estimate because of the challenges of measuring the distribution of stars over space and velocity

– line-of-sight velocity component can be measured from Doppler shift

– but transverse velocity component cannot be measured directly for galaxies beyond the Local Group

So we measure the number density n of stars instead:

– star counts plus estimates of distances for individual stars can provide n as a function of position within our Galaxy

– for a distant galaxy, convert intensity along line of sight of the *integrated light* from large numbers of stars to number densities – 'deprojection' – needs assumption about stellar populations and their three-dimensional distribution. But can be done

Spectroscopy provides mean velocities $\langle v_l \rangle$ and widths of absorption lines provide velocity dispersions σ_l, both along the line of sight. In our own Galaxy, it is usually possible to calculate the mean velocity and dispersion about the mean

Observational data can provide the velocity dispersion in perpendicular directions, at least within our own Galaxy

So in our Galaxy, at each point in space, we might have the mean values of the velocity components, $\langle v_R \rangle$, $\langle v_\phi \rangle$ and $\langle v_z \rangle$, in the R, ϕ and z directions, plus the dispersions σ_R, σ_ϕ and σ_z, of the velocity components about their mean values, expressed as standard deviations

Velocity dispersions often represented by the *velocity dispersion ellipsoid* — idealised representation of the dispersions as a 3D ellipsoid where distance from origin in any particular direction is size of the velocity dispersion in that direction

When working with galaxies other than our own, we sometimes consider *isotropic velocity distributions* (particularly for elliptical galaxies) — in this case, the velocity dispersions in each direction are the same: $\sigma_r = \sigma_\theta = \sigma_\phi$, using a spherical coordinate system (r, θ, ϕ) here

We may abbreviate these equal components simply as σ, and these will be the same as the velocity dispersion σ_l along our line of sight

If the mean velocities are zero i.e. $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$ (as will be the case if the galaxy is in a steady state and has no net rotation) then $\sigma^2 = \langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$.

The mean of the square of the space velocity will be $\langle v^2 \rangle = \langle v_r^2 \rangle + \langle v_\theta^2 \rangle + \langle v_\phi^2 \rangle$. Therefore, $\langle v^2 \rangle = 3\sigma^2$ in a steady-state with no net rotation

Although we often use the velocity dispersions in three perpendicular directions (such as σ_R, σ_ϕ and σ_z), a full description of the dynamics of stars requires a *velocity dispersion tensor* σ_{ij}. See later

It is therefore much more convenient to calculate quantities involving number densities n, mean velocities and velocity dispersions from f

These quantities can then be compared with observations more directly. A series of equations called *The Jeans Equations* allow this to be done

The Jeans Equations

The Jeans Equations relate:

- **number densities** n
- **mean velocities** $\langle v_i \rangle$
- **velocity dispersions** σ_{ij}
- **gravitational potential** Φ

First developed in stellar dynamics by Sir James Jeans in 1919

Some useful quantities relating to the distribution function, f**:**

$$\begin{aligned} n &= \int f \, \mathrm{d}^3\mathbf{v} \\ n \, \langle v_i \rangle &\equiv \int v_i \, f \, \mathrm{d}^3\mathbf{v} \\ n \, \sigma_{ij}^2 &\equiv \int (v_i - \langle v_i \rangle) \, (v_j - \langle v_j \rangle) \, f \, \mathrm{d}^3\mathbf{v} \\ \textbf{(for } i,j &= 1,2,3\textbf{)} \end{aligned}$$

$$\begin{aligned} \textbf{zeroth moment} &= \int x^0 \, f(x) \, \mathrm{d}x \\ \textbf{first moment} &= \int x^1 \, f(x) \, \mathrm{d}x \\ \textbf{second moment} &= \int x^2 \, f(x) \, \mathrm{d}x \end{aligned}$$

The First Jeans Equation

The collisionless Boltzmann equation gives:

$$\frac{\partial f}{\partial t} \;+\; \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} \;+\; \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) \;=\; 0$$

or equivalently

$$\frac{\partial f}{\partial t} \;+\; \sum_{i=1}^{3} v_i\,\frac{\partial f}{\partial x_i} \;-\; \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} \;=\; 0$$

on substituting for the components of acceleration from $\mathbf{dv}/\mathrm{d}t = -\boldsymbol{\nabla}\Phi$

To derive the first Jeans equation, we find the zeroth moment of the CBE:

$$\int\left(\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i\frac{\partial f}{\partial x_i} - \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\right)\mathrm{d}^3\mathbf{v} \;=\; \int 0\,.\,\mathrm{d}^3\mathbf{v}$$

$$\therefore\quad \int\frac{\partial f}{\partial t}\,\mathrm{d}^3\mathbf{v} + \sum_{i=1}^{3}\int v_i\,\frac{\partial f}{\partial x_i}\mathrm{d}^3\mathbf{v} - \sum_{i=1}^{3}\int\frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} \;=\; 0$$

Some terms can be simplified e.g. by noting integration is over all velocities at each position and time:

Also, since t and v_i's are independent coordinates:

$$\int\frac{\partial f}{\partial t}\,\mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial t}\int f\,\mathrm{d}^3\mathbf{v}$$

$$\;=\; \frac{\partial n}{\partial t} \quad \textbf{because } n = \textstyle\int f\,\mathrm{d}^3\mathbf{v}$$

and since v_i's and x_i's are independent coordinates:

$$\begin{aligned}
\int v_i \frac{\partial f}{\partial x_i} \, \mathrm{d}^3\mathbf{v} &= \int \frac{\partial (v_i f)}{\partial x_i} \, \mathrm{d}^3\mathbf{v} \\
&= \frac{\partial}{\partial x_i} \int v_i \, f \, \mathrm{d}^3\mathbf{v} \\
&= \frac{\partial \left(n \left\langle v_i \right\rangle \right)}{\partial x_i}
\end{aligned}$$

on substituting $n\langle v_i\rangle = \int v_i f \mathrm{d}^3\mathbf{v}$

Since x_i's and Φ are independent of v_i's:

$$\begin{aligned}
\int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \, \mathrm{d}^3\mathbf{v} &= \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \, \mathrm{d}^3\mathbf{v} \\
&= \frac{\partial \Phi}{\partial x_i} \, (0)
\end{aligned}$$

because $f \longrightarrow 0$ as $|v_i| \longrightarrow \infty$ (by analogy with the divergence theorem)

$$= \quad 0$$

Substituting for these terms:

$$\boxed{\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial n \langle v_i \rangle}{\partial x_i} = 0}$$

which is a continuity equation

This is the First Jeans Equation

The Second Jeans Equation

To derive the second of the Jeans Equations, we find the first moment of the CBE:

Writing the CBE as a summation over j we get

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{3} v_j \frac{\partial f}{\partial x_j} - \sum_{j=1}^{3} \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} = 0$$

Multiply by v_i, where $i = 1, 2$ or 3:

$$v_i \frac{\partial f}{\partial t} + v_i \sum_{j=1}^{3} v_j \frac{\partial f}{\partial x_j} - v_i \sum_{j=1}^{3} \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} = 0$$

Integrate over all velocities:

$$\int \left(v_i \frac{\partial f}{\partial t} + \sum_{j=1}^{3} v_i \, v_j \frac{\partial f}{\partial x_j} - \sum_{j=1}^{3} v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} \right) \mathrm{d}^3\mathbf{v} = \int 0.\mathrm{d}^3\mathbf{v}$$

Therefore:

$$\int v_i \frac{\partial f}{\partial t} \, \mathrm{d}^3\mathbf{v} + \sum_{j=1}^{3} \int v_i v_j \frac{\partial f}{\partial x_j} \, \mathrm{d}^3\mathbf{v} - \sum_{j=1}^{3} \int v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} = 0$$

(where $i = 1, 2$ or 3)

In the first term, because v_i and t are independent coordinates, we can use:

$$\begin{aligned}\int v_i \frac{\partial f}{\partial t}\, \mathrm{d}^3\mathbf{v} &= \int \frac{\partial (v_i\, f)}{\partial t}\, \mathrm{d}^3\mathbf{v} \\ &= \frac{\partial}{\partial t} \int v_i\, f\, \mathrm{d}^3\mathbf{v} \\ &= \frac{\partial}{\partial t}(n\, \langle v_i \rangle) \quad \textbf{because } n\,\langle v_i\rangle = \textstyle\int v_i\, f\, \mathrm{d}^3\mathbf{v}\end{aligned}$$

and in the second term, because v_i and v_j are independent of x_j:

$$\int v_i\, v_j\, \frac{\partial f}{\partial x_j}\, \mathrm{d}^3\mathbf{v} = \int \frac{\partial}{\partial x_j}(v_i\, v_j\, f)\, \mathrm{d}^3\mathbf{v}$$

also because x_j and v_i's are independent coordinates:

$$= \frac{\partial}{\partial x_j} \int v_i v_j f\, \mathrm{d}^3\mathbf{v}$$

and on substituting $n\langle v_i v_j\rangle \equiv \int v_i v_j f\, \mathrm{d}^3\mathbf{v}$:

$$= \frac{\partial\,(n\,\langle v_i v_j\rangle\,)}{\partial x_j}$$

In the third term, x_j's and Φ are independent of v_i's and so:

$$\int v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\mathbf{v} = \frac{\partial \Phi}{\partial x_j} \int v_i \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\mathbf{v}$$

also, the product rule gives:

$$\frac{\partial (v_i f)}{\partial v_j} = v_i \frac{\partial f}{\partial v_j} + f \frac{\partial v_i}{\partial v_j}$$

therefore

$$v_i \frac{\partial f}{\partial v_j} = \frac{\partial (v_i f)}{\partial v_j} - f \frac{\partial v_i}{\partial v_j}$$

and since v_i and v_j are independent if $i \neq j$:

$$\begin{aligned} \frac{\partial v_i}{\partial v_j} &= 1 \quad \textbf{if } i = j \\ &= 0 \quad \textbf{if } i \neq j \end{aligned}$$

or equivalently:

$$\begin{aligned} \frac{\partial v_i}{\partial v_j} &= \delta_{ij} \\ \therefore \quad v_i \frac{\partial f}{\partial v_j} &= \frac{\partial (v_i f)}{\partial v_j} - \delta_{ij} f \end{aligned}$$

$$\begin{aligned} \textbf{So} \quad \int v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} &= \frac{\partial \Phi}{\partial x_j} \int \left(\frac{\partial (v_i f)}{\partial v_j} - \delta_{ij} f \right) \mathrm{d}^3\mathbf{v} \\ &= \frac{\partial \Phi}{\partial x_j} \left(\int \frac{\partial (v_i f)}{\partial v_j} \, \mathrm{d}^3\mathbf{v} - \delta_{ij} \int f \, \mathrm{d}^3\mathbf{v} \right) \end{aligned}$$

and because $v_i f \longrightarrow 0$ as $|v_i| \longrightarrow \infty$:

$$\begin{aligned} &= \frac{\partial \Phi}{\partial x_j} \left(0 - \delta_{ij} \, n \right) \\ &= -\frac{\partial \Phi}{\partial x_j} \delta_{ij} \, n \end{aligned}$$

Substituting for these terms:

$$\frac{\partial (n \langle v_i \rangle)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \langle v_i v_j \rangle \big) - \sum_{j=1}^{3} \left(-\frac{\partial \Phi}{\partial x_j} \delta_{ij} \, n \right) = 0$$

So finally we get:

$$\boxed{\frac{\partial(n\langle v_i\rangle)}{\partial t} \;+\; \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\big(n\langle v_i v_j\rangle\big) \;=\; -\,\frac{\partial \Phi}{\partial x_i}\, n}$$

for each of $i = 1, 2, 3$

This is the Second Jeans Equation