SPA7010 Example Problems 4
(20 Feb 2020)

Stellar Dynamics

Problem 4.1

(a) Define the terms relaxation time, $T_{\rm relax}$, and crossing time, $T_{\rm cross}$, in the context of stellar dynamics.

(b) A star moves through a spherical region, radius R, containing N other stars, each of mass m, distributed uniformly in space. The total change in v^2, the square of the star's velocity, in time t, caused by weak gravitational encounters with other stars, can be expressed as

$$\Delta v^2(t) \quad = \quad 6 \left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right) ,$$

where $b_{\rm max}$ and $b_{\rm min}$ are the largest and smallest values of the impact parameter.

Write down an expression for $T_{\rm relax}$ using the expression given above.

(c) Show that for suitable choices of $b_{\rm min}$ and $b_{\rm max}$, the ratio of the relaxation time to the crossing time, $T_{\rm cross}$, is given approximately by

$$\frac{T_{\rm relax}}{T_{\rm cross}} \approx \frac{N}{12 \ln N} .$$

(You may assume that in a stellar system of radius R containing N stars each of mass m, the typical velocity v is given by $v \approx \sqrt{GNm/R}$.)

(d) The region around the nucleus of a galaxy is observed to consist of a dense cluster of 10^7 stars moving in randomly orientated orbits with typical velocities of $100\,{\rm km\,s^{-1}}$. The radius of this region is observed to be 70 pc.

Assuming the density of stars is uniform across the cluster, estimate the relaxation time of the stellar motions.

How does the relaxation time compare with the age of the galaxy? Can the dynamics of the stars around the nucleus be modelled as a collisionless system over the lifetime of the galaxy? How does this compare with the stars in regions away from the nucleus?

[The constant of gravitation is $G = 6.673 \times 10^{-11}\ {\rm m^3\,kg^{-1}s^{-2}}$. The mass of the Sun is $1\ M_\odot = 1.989 \times 10^{30}$ kg. One year is 3.1557×10^7 s. The age of the Universe is about 13.7 Gyr.]

Problem 4.2

(a) A spherical galaxy has a uniform density, a total mass M and a radius R. Using the virial theorem, show that the stellar velocity, v, is related to the mass and radius by

$$v^2 = \frac{3}{5} \frac{GM}{R} ,$$

if the internal gravitational potential energy of a uniform sphere of mass M and radius R is

$$V = -\frac{3}{5} \frac{GM^2}{R} .$$

(b) Would you expect the virial theorem result $2\langle T\rangle + \langle V\rangle = 0$ to apply collectively to the galaxies in a cluster of galaxies that is still collapsing gravitationally?

(c) The virial theorem is often applied to hot gas in clusters of galaxies. What are the particles in this case?

(d) A globular cluster is represented by a Plummer potential with a softening parameter $a = 2 \times 10^{17}\,\mathrm{m}$. Observations show that the root-mean-square velocity of stars in the cluster is $v = 10\,\mathrm{km\,s^{-1}}$. The total internal potential energy for this potential is

$$V = -\frac{3\pi}{32} \frac{G\,M_{tot}^2}{a} ,$$

where M_{tot} is the total mass of the cluster and G is the constant of gravitation.

Estimate the total mass of the cluster, giving your answer in solar masses.

(The Plummer model has a gravitational potential $\Phi(r)$ at a distance r given by $\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2+a^2}}$, where M_{tot} is the total mass, G is the constant of gravitation, and a is a constant. $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$. The solar mass is $M_\odot = 1.99 \times 10^{30}$ kg.)