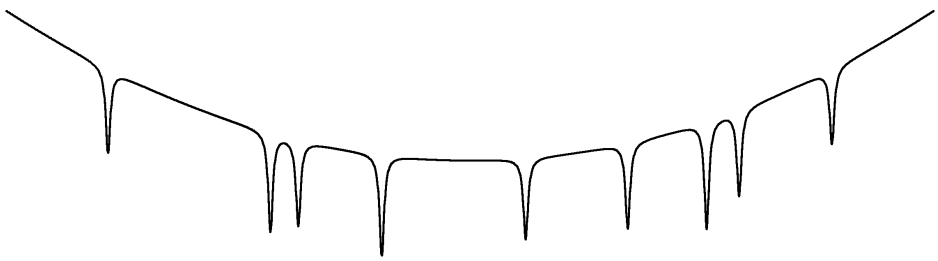
The nature of the gravitational potential in a galaxy

The gravitational potential in a galaxy essentially has two components:

- broad, smooth, underlying potential due to the entire galaxy

- the localised deeper potentials due to individual stars



It is the broad, smooth component that determines the motions of stars

So we can represent the dynamics of a system of stars using only the smooth underlying component of the gravitational potential $\Phi(\mathbf{x}, t)$, where x is the position vector of a point and t is the time

If the galaxy has reached a steady state, Φ is $\Phi(\mathbf{x})$ only

Gravitational potentials, density distributions and masses

General principles

Distribution of mass in a galaxy – both visible and dark matter – determines the gravitational potential

The potential Φ at any point is related to the local density ρ by Poisson's Equation:

$$\nabla^2\Phi \; (\equiv \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{\nabla}\Phi) \; = \; 4\pi\, G\rho$$

So if we know $\rho(\mathbf{x})$ across a galaxy, we can calculate the potential Φ, either analytically or numerically, by integration

Alternatively, if we know $\Phi(\mathbf{x})$, we can calculate the density profile $\rho(\mathbf{x})$ by differentiation. In addition, because the acceleration due to gravity $\mathbf{g}$ is related to the potential by $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, we can compute $\mathbf{g}(\mathbf{x})$ from $\Phi(\mathbf{x})$ and vice-versa. Similarly, substituting for $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ in the Poisson Equation gives $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{g} = -\,4\pi G\rho$

These computations are often done for some typical theoretical representations of the potential or density. A number of convenient analytical functions are encountered in the literature, depending on the type of galaxy being modelled and particular circumstances

The issue of determining actual density profiles and potentials from observations of galaxies is much more challenging

Observations readily give the projected density distributions of stars on the sky, and we can attempt to derive the three-dimensional distribution of stars from this; this in turn can give the density of visible matter $\rho_{VIS}(\mathbf{x})$ across the galaxy

However, it is the total density $\rho(\mathbf{x})$, including dark matter $\rho_{DM}(\mathbf{x})$, that is relevant gravitationally, with $\rho(\mathbf{x}) = \rho_{DM}(\mathbf{x}) + \rho_{VIS}(\mathbf{x})$

The dark matter distribution can only be inferred from the dynamics of visible matter (or to a limited extent from gravitational lensing of background objects)

So in practice, the 3D density distribution $\rho(\mathbf{x})$ and the gravitational potential $\Phi(\mathbf{x})$ are poorly known, particularly where dark matter dominates far from the central regions

Spherical symmetry

Calculating the relationship between density and potential is much simpler for spherically symmetric distributions – for example spherical elliptical galaxies

Under spherical symmetry, ρ and Φ are functions only of the radial distance r from the centre of the distribution. Therefore,

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \;=\; 4\,\pi\,G\,\rho \qquad (1)$$

because Φ is independent of the angles θ and ϕ in a spherical coordinate system (see Appendix C in the Course Notes)

Also, for spherically symmetric distributions, the mass $M(r)$ that lies inside a radius r can be related to the density $\rho(r)$ by considering a thin spherical shell of radius r and thickness dr centred on the distribution

The mass of this shell is d$M(r) = \rho(r) \times$surface area$\times$ thickness $= 4\pi r^2\rho(r)\,\mathrm{d}r$. This gives us the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}r} \;=\; 4\pi\,r^2\,\rho \qquad (2)$$

often known as the equation of continuity of mass

The total mass is $M_{tot} = \lim_{r\to\infty} M(r)$

The gravitational acceleration, g, in a spherical distribution has an absolute value |g| of

$$g = \frac{GM(r)}{r^2}$$

at a distance r from the centre, where G is the constant of gravitation (Appendix B), directed towards the centre of the distribution

If we know how one of these functions (ρ, Φ or $M(r)$) depends on radial distance r, we can calculate the others relatively easily when we have spherical symmetry

For example, if know the potential $\Phi(r)$ as a function of r, we can differentiate it to get the mass $M(r)$ interior to r, and by differentiating it again we can get the density $\rho(r)$

Or, if we know $\rho(r)$ as a function of r, we can integrate it to get $M(r)$, and integrating it again gives $\Phi(r)$

Comparing equations (1) and (2) we find :

$$M(r) = \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r}$$

when we have spherical symmetry. This allows us to convert between $M(r)$ and $\Phi(r)$ directly for this spherically symmetric case.

Three examples of spherical potentials

(1) The Plummer Potential

Often used for the theoretical modelling of spherically-symmetric galaxies

Has a gravitational potential Φ at a radial distance r from the centre given by :

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2 + a^2}}$$

where M_{tot} is the total mass of the galaxy and a is a constant

The constant a can be used to flatten the potential in the core

For this potential the density ρ at a radial distance r is given by :

$$\rho(r) = \frac{3M_{tot}}{4\pi} \frac{a^2}{(r^2 + a^2)^{5/2}}$$

This can be derived from the expression for Φ using the Poisson equation $\nabla^2\Phi = 4\pi G\rho$

Note: this density scales with radius as $\rho \sim r^{-5}$ at large radii

The mass interior to a point $M(r)$ can be computed from the density ρ using d$M/\mathrm{d}r = 4\pi r^2\rho$, or from the potential Φ using Gauss's Law in the form $\int_S \boldsymbol{\nabla}\Phi\boldsymbol{.}\mathrm{d}\boldsymbol{S} = 4\pi GM(r)$ for a spherical surface of radius r

The result is :

$$M(r) \;=\; \frac{M_{tot}\,r^3}{(r^2+a^2)^{3/2}}$$

The Plummer potential was first used in 1911 by H. C. K. Plummer (1875–1946) to describe globular clusters

Its simple functional form makes it useful for approximate analytical modelling of galaxies but the r^{-5} density profile is much steeper than elliptical galaxies are observed to have

(2) The Dark Matter Profile

In this model, the total density is given by :

$$\rho(r) \;=\; \frac{\rho_0}{1 \,+\, (r/a)^2} \;=\; \frac{\rho_0\, a^2}{r^2 \,+\, a^2}$$

where ρ_0 is the central density ($\rho(r)$ at $r = 0$) and a is a constant

The mass interior to a radius r is

$$M(r) \;=\; 4\pi\rho_0 \int_0^r \frac{r'^2}{1 + r'^2/a^2}\, \mathrm{d}r' \;=\; 4\pi\rho_0 a^2\,(\,r - a\tan^{-1}(r/a)\,)$$

Spiral galaxies with this profile would have rotation curves that are flat for $r \gg a$, which is exactly what is observed

This model therefore successfully matches the large amount of dark matter observed at large distances r from the centres of galaxies

Disadvantage is that the mass interior to a radius tends to infinity as r increases: $\lim_{r\to\infty} M(r) \to \infty$

Also, in practice, density profiles of real galaxies must fall below the dark matter profile at some very large distances

(3) The Isothermal Sphere

This model is identical to the distribution that would be followed by a stable cloud of gas having the same temperature everywhere

A spherically-symmetric cloud of gas having a single temperature T throughout would have a gas pressure $P(r)$ at a radius r from its centre that is related to T by the ideal gas law as $P(r) = n_p k_B T$, where $n_p(r)$ is the number density of gas particles (atoms or molecules) at radius r and k_B is the Boltzmann constant

This can also be expressed in terms of the density ρ as $P(r) = k_B \rho(r) T / m_p$, where m_p is the mean mass of each particle in the gas

The cloud will be supported by hydrostatic equilibrium, so therefore

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\,\rho(r)$$

where $M(r)$ is the mass enclosed within a radius r

The gradient in the mass is $\mathrm{d}M/\mathrm{d}r = 4\pi r^2 \rho(r)$

These equations have a solution

$$\rho(r) \;=\; \frac{\sigma^2}{2\pi G\, r^2} \quad \textbf{and} \quad M(r) \;=\; \frac{2\sigma^2}{G}\, r$$

where

$$\sigma^2 \;\equiv\; \frac{k_B T}{m_p}$$

and m_p is the mass of each gas particle

The parameter σ is the root-mean-square velocity in any direction

This is only one of a number of solutions and it is called the *singular isothermal sphere*

The isothermal sphere model for a system of stars is defined to be a model that has the same density distribution as an isothermal gas cloud

So an isothermal galaxy would also have a density $\rho(r)$ and mass $M(r)$ interior to a radius r given by

$$\rho(r) \;=\; \frac{\sigma^2}{2\pi G\, r^2} \quad \textbf{and} \quad M(r) \;=\; \frac{2\sigma^2}{G}\, r$$

for a singular isothermal sphere, where σ is root-mean-square velocity of the stars along any direction

This model is simple but also unrealistic in some important respects

Most significantly, the model fails totally at large radii since the limit of $M(r)$ as $r \longrightarrow \infty$ is infinite

Phase space and the distribution function $f(x, v, t)$

To describe the dynamics of a galaxy, we could use:

- **the positions of each star, $\mathbf{x}_i$**
- **the velocities of each star, $\mathbf{v}_i$**

where $i = 1$ to N, with $N \sim 10^6$ to 10^{12} - requires an enormous database

In practice, therefore, we represent the stars in a galaxy using a *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$

- the probability density in the 6-dimensional phase space of position and velocity at a given time

- also known as the "phase space density"

The *number* of stars in a rectangular box between x and $x + \mathrm{d}x$, y and $y + \mathrm{d}y$, z and $z + \mathrm{d}z$, with velocity components between v_x and $v_x + \mathrm{d}v_x$, v_y and $v_y + \mathrm{d}v_y$, v_z and $v_z + \mathrm{d}v_z$, is

$$f(\mathbf{x}, \mathbf{v}, t)\; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z \;\equiv\; f(\mathbf{x}, \mathbf{v}, t)\; \mathrm{d}^3\mathbf{x}\, \mathrm{d}^3\mathbf{v}$$

The *number density*, $n(\mathbf{x}, t)$, of stars is obtained from the distribution function f by integrating over the *velocity* components:

$$\begin{aligned} n(\mathbf{x}, t) &= \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z \\ &= \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\, \mathrm{d}^3\mathbf{v} \end{aligned}$$

Note that:

The units of f are :

(Number of stars) / (pc^3) / (kms^{-1})^3

The units of *number density* $n(\mathbf{x}, t)$ are:

(Number of stars) / (pc^3)

The Continuity equation

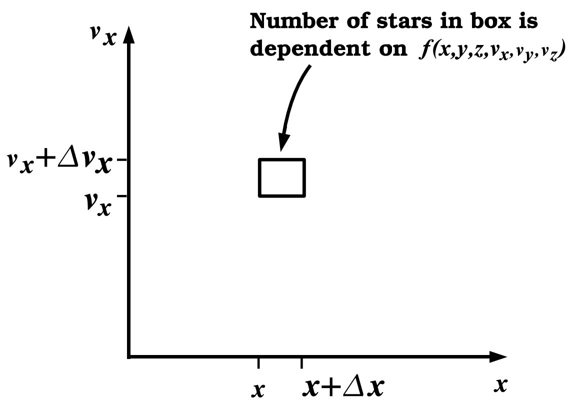
Let's assume that stars are conserved i.e. let's ignore star formation, supernovae etc.

This leads to the *continuity equation*

It expresses the rate of change in the distribution function f as a function of time to the rates of change with position and velocity

Consider the $x - v_x$ plane within the 6-D phase space (x, y, z, v_x, v_y, v_z) in Cartesian coordinates

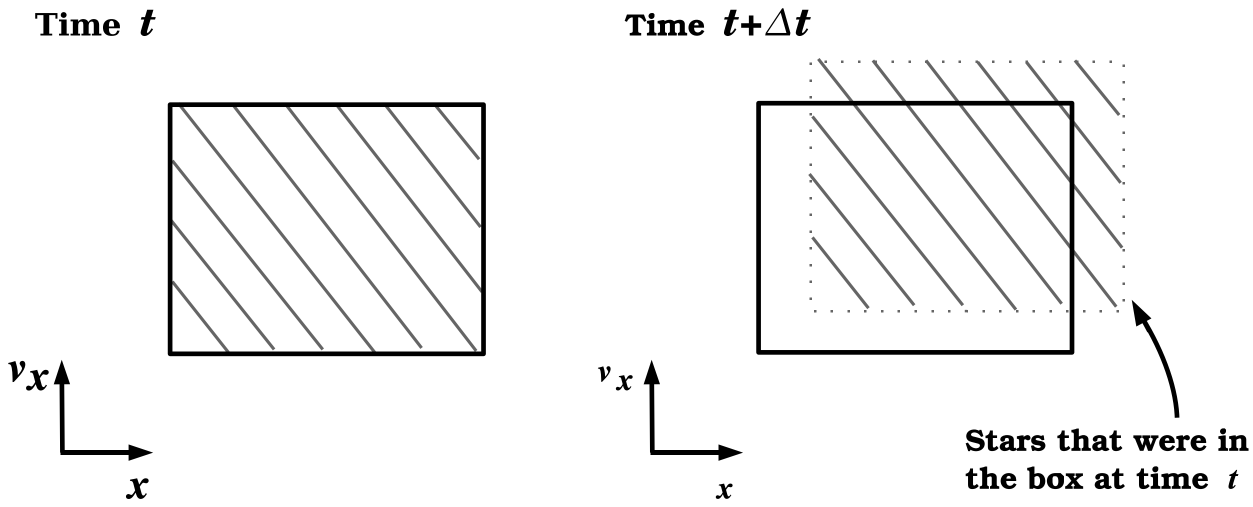
Consider a rectangular box in this plane extending from x to $x + \Delta x$ and v_x to $v_x + \Delta v_x$



The velocity v_x means that stars move in x ($v_x \equiv \mathrm{d}x/\mathrm{d}t$)

So there is a flow of stars through the box in both the x and the v_x directions

The box in more detail:



We can represent the flow of stars by the continuity equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(f\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{\partial}{\partial y}\left(f\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \frac{\partial}{\partial z}\left(f\frac{\mathrm{d}z}{\mathrm{d}t}\right) +$$

$$\frac{\partial}{\partial v_x}\left(f\frac{\mathrm{d}v_x}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_y}\left(f\frac{\mathrm{d}v_y}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_z}\left(f\frac{\mathrm{d}v_z}{\mathrm{d}t}\right) = 0$$

This can be abbreviated as

$$\boxed{\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\left(f\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_i}\left(f\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)\right) = 0}$$

where $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $v_1 \equiv v_x$, $v_2 \equiv v_y$, **and** $v_3 \equiv v_z$

The continuity equation is sometimes also abbreviated as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \bullet \left(f \, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \right) + \frac{\partial}{\partial \mathbf{v}} \bullet \left(f \, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \right) = 0$$

where, in this notation, for any vectors a and b with components (a_1, a_2, a_3) **and** (b_1, b_2, b_3)**,**

$$\frac{\partial}{\partial \mathbf{a}} \bullet \mathbf{b} \equiv \sum_{i=1}^{3} \frac{\partial b_i}{\partial a_i}$$

(not a direct differentiation by a vector)

We can simplify the notation further using a combined phase space coordinate system $\mathbf{w} = (\mathbf{x}, \mathbf{v})$ **with components** $(w_1, w_2, w_3, w_4, w_5, w_6) = (x, y, z, v_x, v_y, v_z)$

The continuity equation then becomes

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial}{\partial w_i} (f \dot{w}_i) = 0$$

The continuity equation can also be expressed in terms of the momentum $\mathbf{p} = m\mathbf{v}$**, where** m **is mass of an element of gas, as**

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \bullet \left(f \, \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \right) + \frac{\partial}{\partial \mathbf{p}} \bullet \left(f \, \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \right) = 0$$

The Collisionless Boltzmann Equation

We know that T_{relax} is very long for galaxies - stars in galaxies are essentially collisionless

It is possible to derive an equation from the continuity equation that more explicitly states the relation between the distribution function f, position x, velocity v and time t

This is the collisionless Boltzmann equation

It states that :

$$\boxed{\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = 0}$$

Derivation of the Collisionless Boltzmann Equation

Continuity equation states

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i} \left(f \, \frac{\mathrm{d}x_i}{\mathrm{d}t} \right) + \frac{\partial}{\partial v_i} \left(f \, \frac{\mathrm{d}v_i}{\mathrm{d}t} \right) \right) = 0$$

where f is the distribution function in the cartesian phase space $(x_1, x_2, x_3, v_1, v_2, v_3)$

The acceleration of a star is the gradient of its gravitational potential Φ:

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = - \frac{\partial \Phi}{\partial x_i}$$

in each direction (i.e. for each value of i for $i = 1, 2, 3$)

This is simply $\mathbf{dv}/\mathbf{d}t = \mathbf{g} = -\boldsymbol{\nabla}\Phi$ resolved into each dimension

We also have $\dfrac{\mathrm{d}x_i}{\mathrm{d}t} = v_i$, so,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial}{\partial x_i} (f v_i) + \frac{\partial}{\partial v_i} \left(-f \frac{\partial \Phi}{\partial x_i} \right) \right) = 0$$

But v_i is a coordinate, not a value associated with a particular star: we are using the continuous function f rather than considering individual stars

So v_i is independent of x_i and

$$\frac{\partial}{\partial x_i}(f v_i) \quad = \quad v_i \frac{\partial f}{\partial x_i}$$

The potential $\Phi \equiv \Phi(\mathbf{x}, t)$ does not depend on v_i: Φ is independent of velocity

$$\therefore \quad \frac{\partial}{\partial v_i}\left(f \frac{\partial \Phi}{\partial x_i}\right) \quad = \quad \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}$$

$$\therefore \quad \frac{\partial f}{\partial t} \; + \; \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial x_i} \; - \; \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} \right) \quad = \quad 0$$

But $\dfrac{\mathrm{d}v_i}{\mathrm{d}t} \; = \; -\dfrac{\partial \Phi}{\partial x_i}$, **so,**

$$\frac{\partial f}{\partial t} \; + \; \sum_{i=1}^{3}\left(v_i \frac{\partial f}{\partial x_i} \; + \; \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i} \right) \quad = \quad 0$$

This is the collisionless Boltzmann equation

So,

$$\frac{\partial f}{\partial t} \; + \; \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} \; + \; \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i} \right) \quad = \quad 0$$

The left-hand side is the finite differential $\mathrm{d}f/\mathrm{d}t$. So the CBE can also be written as

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} \quad \equiv \quad \frac{\partial f}{\partial t} \; + \; \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} \; + \; \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i} \right) \quad = \quad 0}$$

Alternatively it can expressed as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \;\equiv\; \frac{\partial f}{\partial t} \;+\; \sum_{i=1}^{6} \dot{w}_i \, \frac{\partial f}{\partial w_i} \;=\; 0$$

where $\mathbf{w} = (\mathbf{x}, \mathbf{v})$ is a 6-dimensional coordinate system, and also as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \;\equiv\; \frac{\partial f}{\partial t} \;+\; \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} \;+\; \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{v}} \;=\; 0$$

and as

$$\frac{\mathrm{d}f}{\mathrm{d}t} \;\equiv\; \frac{\partial f}{\partial t} \;+\; \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} \;+\; \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{p}} \;=\; 0$$

Note the use here of the notation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} \;\equiv\; \sum_{i=1}^{3} \frac{\mathrm{d}x_i}{\mathrm{d}t}\,\frac{\partial f}{\partial x_i} \qquad \textbf{etc.}$$

The implications of the Collisionless Boltzmann Equation

The collisionless Boltzmann equation (CBE) implies

$$\mathrm{d}f/\mathrm{d}t = 0$$

i.e. the *local* density in phase space, f, does not change with time along a star's orbit

So if we follow a star in orbit, the density f in 6-D phase space around the star is constant

If a star moves inwards in a galaxy along its orbit, the *spatial* density of stars increases

$\mathrm{d}f/\mathrm{d}t = 0$ **then implies that the *velocity* density around the star will decrease to keep f constant**

$\therefore$ **the velocity *dispersion* around the star *increases* as the star moves inwards**

The CBE and the Poisson equation together constitute the basic equations of stellar dynamics:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \ , \qquad \nabla^2\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x})$$

where f is the distribution function, t is time, $\Phi(\mathbf{x}, t)$ is the gravitational potential at point x and $\rho(\mathbf{x}, t)$ is the mass density at point x

The collisionless Boltzmann equation applies because star-star encounters do not change the motions of stars significantly over the lifetime of a galaxy

If the system were collisional, the CBE would have to be modified by adding a "collisional term" on the right-hand-side.

The Collisionless Boltzmann Equation in cylindrical coordinates

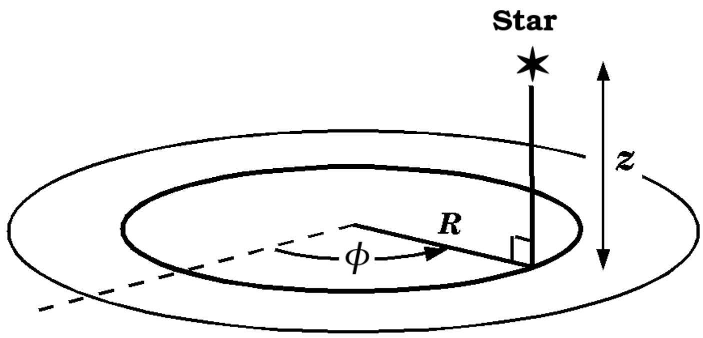
[This section is not examinable]

So far we have considered Cartesian coordinates (x, y, z, v_x, v_y, v_z). However, the form

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0$$

for the collisionless Boltzmann equation applies to any coordinate system

For a galaxy, it is often more convenient to use cylindrical coordinates with the centre of the galaxy as the origin



The coordinates of a star are (R, ϕ, z) and $R = \sqrt{(x^2 + y^2)}$

Cylindrical coordinates particularly useful for spiral galaxies where $z = 0$ plane is the Galactic plane (note ϕ is a coordinate angle, whereas Φ is gravitational potential)

The collisionless Boltzmann equation in this system is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}R}{\mathrm{d}t}\frac{\partial f}{\partial R} + \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\partial f}{\partial \phi} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial f}{\partial z} + \frac{\mathrm{d}v_R}{\mathrm{d}t}\frac{\partial f}{\partial v_R} + \frac{\mathrm{d}v_\phi}{\mathrm{d}t}\frac{\partial f}{\partial v_\phi} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\frac{\partial f}{\partial v_z} = 0$$

where v_R, v_ϕ, and v_z are the components of the velocity in the R, ϕ, z directions

Need to replace the differentials of the velocity components $\mathrm{d}v_R/\mathrm{d}t$, $\mathrm{d}v_\phi/\mathrm{d}t$ and $\mathrm{d}v_z/\mathrm{d}t$ with more convenient terms

Velocity and acceleration in cylindrical coordinates:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t}\hat{\mathbf{e}}_R + R\frac{\mathrm{d}\phi}{\mathrm{d}t}\hat{\mathbf{e}}_\phi + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_z$$

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right)\hat{\mathbf{e}}_R + + \left(2\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\right)\hat{\mathbf{e}}_\phi \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\hat{\mathbf{e}}_z$$

where $\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ are unit vectors in the R, ϕ and z directions (standard result for any cylindrical coordinate system)

Representing the velocity as $\mathbf{v} = v_R\hat{\mathbf{e}}_R + v_\phi\hat{\mathbf{e}}_\phi + v_z\hat{\mathbf{e}}_z$ and equating coefficients

$$\frac{\mathrm{d}R}{\mathrm{d}t} = v_R \;, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v_\phi}{R} \;, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = v_z$$

The acceleration can be related to the gravitational potential Φ with $\mathbf{a} = -\boldsymbol{\nabla}\Phi$. In a cylindrical coordinate system:

$$\boldsymbol{\nabla} \;\equiv\; \hat{\mathbf{e}}_R \frac{\partial}{\partial R} \;+\; \hat{\mathbf{e}}_\phi \frac{1}{R}\frac{\partial}{\partial \phi} \;+\; \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

Using this result and equating coefficients:

$$\begin{aligned}
\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 &= -\frac{\partial\Phi}{\partial R} \\
2\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} &= -\frac{1}{R}\,\frac{\partial\Phi}{\partial\phi} \\
\frac{\mathrm{d}^2z}{\mathrm{d}t^2} &= -\frac{\mathrm{d}\Phi}{\mathrm{d}z}
\end{aligned}$$

Rearranging these and substituting for $\mathrm{d}R/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$ and $\mathrm{d}z/\mathrm{d}t$ we obtain:

$$\begin{aligned}
\frac{\mathrm{d}v_R}{\mathrm{d}t} &= -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R} \\
\frac{\mathrm{d}v_\phi}{\mathrm{d}t} &= -\frac{1}{R}\frac{\partial\Phi}{\partial\phi} - \frac{v_R v_\phi}{R} \\
\textbf{and} \quad \frac{\mathrm{d}v_z}{\mathrm{d}t} &= -\frac{\partial\Phi}{\partial z}
\end{aligned}$$

Substituting these into the CBE we obtain:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0$$

This is the collisionless Boltzmann equation in cylindrical coordinates

It relates f to observable quantities $(R, \phi, z, v_R, v_\phi, v_z)$ and the potential Φ

In many practical cases, particularly spiral galaxies, Φ will be independent of ϕ, so $\partial\Phi/\partial\phi = 0$ (but not if we include spiral arms where the potential will be slightly deeper)

Orbits of stars in galaxies

Orbits: the trajectories of stars within gravitational potentials. Orbits within galaxies are generally:

- **complex**
- **not necessarily closed paths**
- **3D (not in a plane)**
- **highly chaotic (true even if galaxy is in equilibrium)**

They are generally very different to Keplerian orbits

Typical orbit in a spherical potential

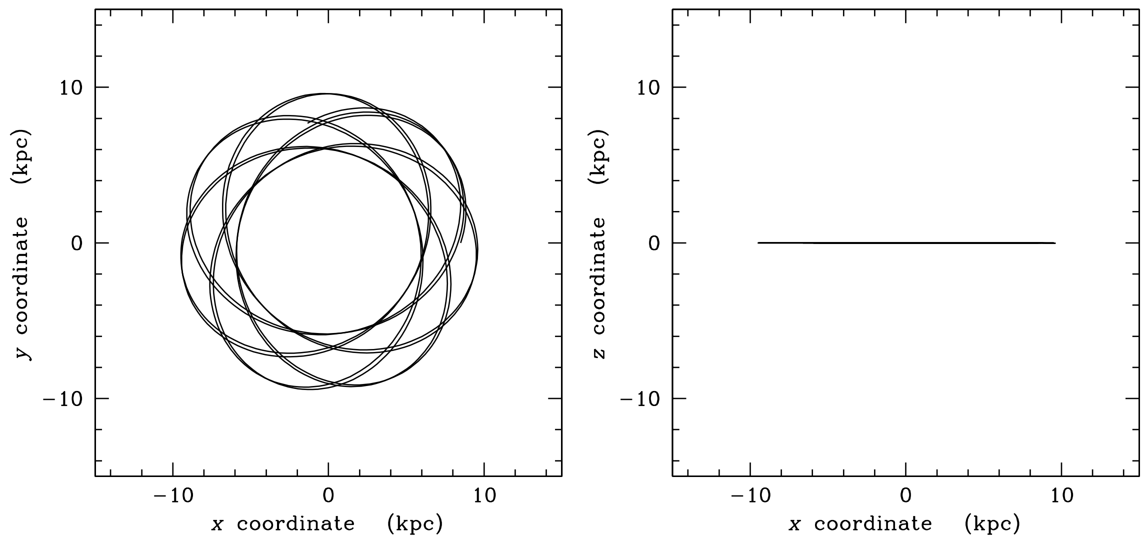


Figure 1: An example of the orbit of a star in a spherical potential. On the left is a view from above, looking down on the $x - y$ plane. On the right is a view from the side, looking edge-on to the $x - y$ plane.

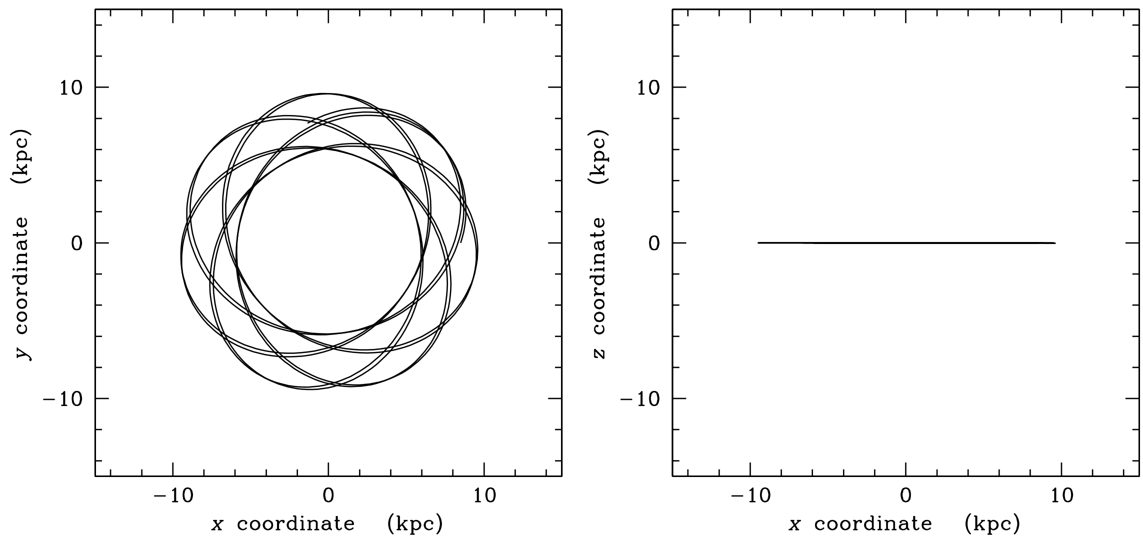
In this example, a star has been put into an orbit in the $x - y$ plane

- **In a spherical potential i.e. where $\Phi = \Phi(r)$ only, the direction of the total angular momentum vector is constant, and so the star's orbit remains confined to the plane perpendicular to the angular momentum vector**

- **The orbit orbit follows a "rosette" pattern when viewed from above, but remains confined to the $x - y$ plane**

[These diagrams were plotted using data generated assuming a Plummer potential: the potential lacks a deep central cusp]

Typical orbit in a spherical potential



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Typical orbit in a triaxial potential

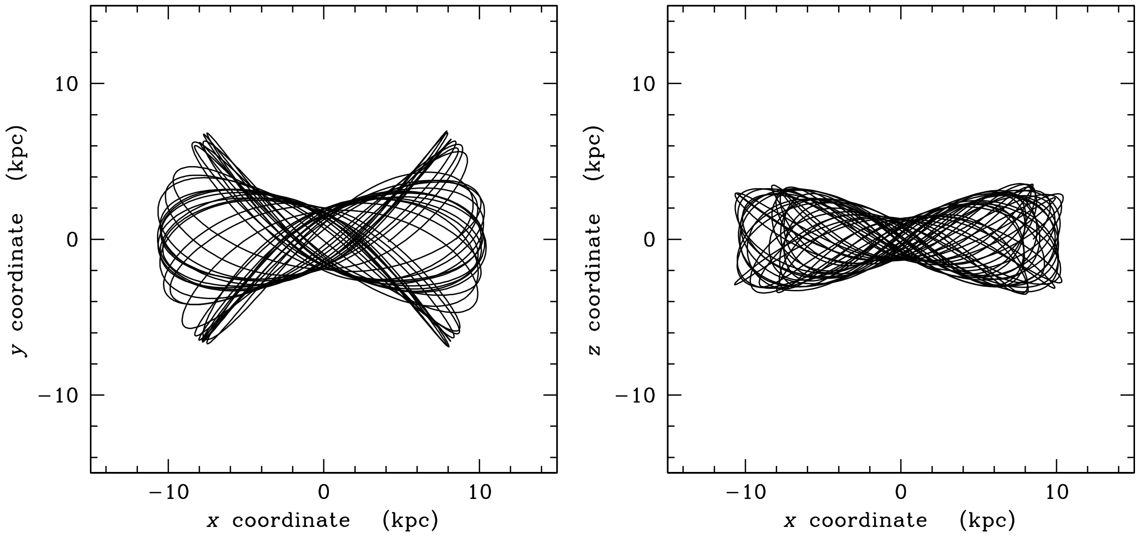

Figure 3: The orbit of a star in a triaxial potential. On the left is a view from above, looking down on the $x - y$ plane. On the right is a view from the side, looking edge-on to the $x - y$ plane.

An example star has been put into an orbit inclined to the $x - y$ plane. The galaxy has different dimensions (different scale sizes) in each of the x, y and z directions (i.e. it is *triaxial*)

- **There is no conserved component of angular momentum**
- **The orbit is complex and 'fills' a volume of space – it can pass very close to the centre**

This illustrates the trajectory of a star in a triaxial elliptical galaxy, for example

(This simulation extends over a longer time period than those in the previous two examples)

Typical orbit in a spherical potential

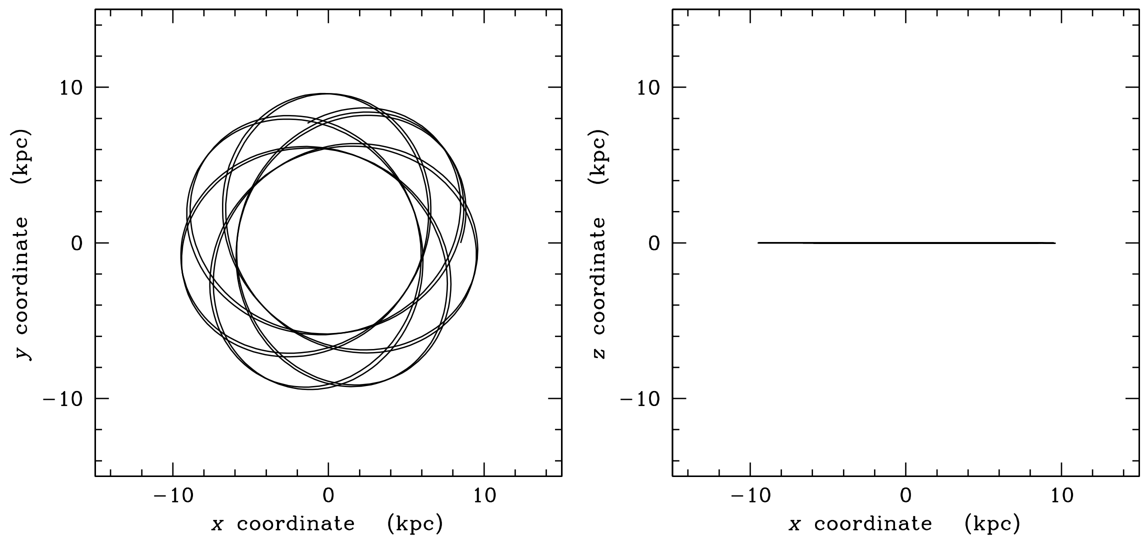

Figure 1: An example of the orbit of a star in a spherical potential. On the left is a view from above, looking down on the $x - y$ plane. On the right is a view from the side, looking edge-on to the $x - y$ plane.

In this example, a star has been put into an orbit in the $x - y$ plane

- **In a spherical potential i.e. where $\Phi = \Phi(r)$ only, the direction of the total angular momentum vector is constant, and so the star's orbit remains confined to the plane perpendicular to the angular momentum vector**

- **The orbit orbit follows a "rosette" pattern when viewed from above, but remains confined to the $x - y$ plane**

[These diagrams were plotted using data generated assuming a Plummer potential: the potential lacks a deep central cusp]

Typical orbit in an oblate potential

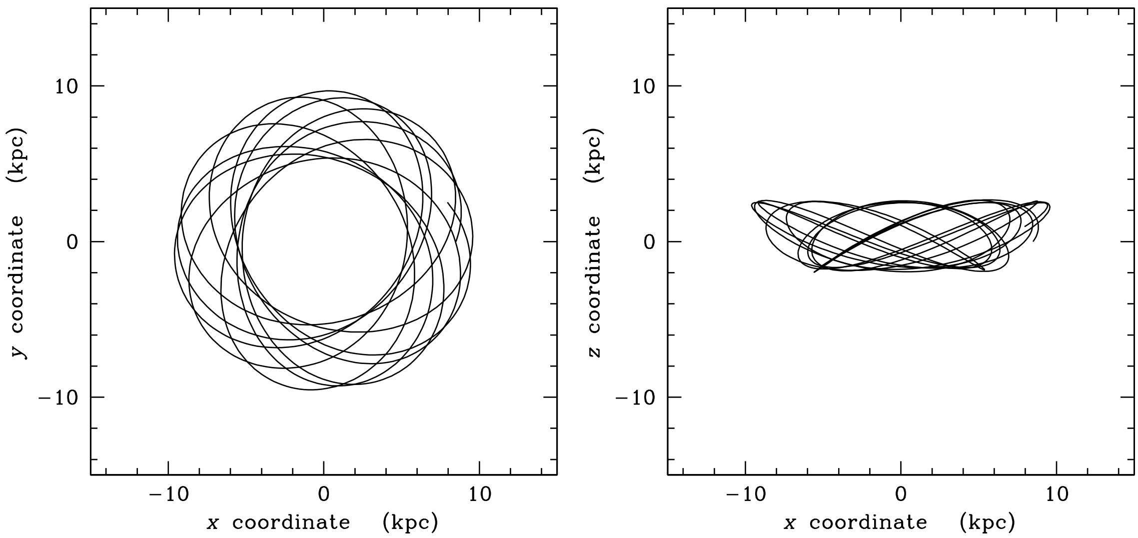

Figure 2: The orbit of a star in a flattened (oblate) potential. On the left is a view from above, looking down on the $x - y$ plane. On the right is a view from the side, looking edge-on to the $x - y$ plane.

Here, an example star has been put into an orbit inclined to the $x - y$ plane. The galaxy is flattened (oblate) in the z direction with an axis ratio of 0.7

- **This potential is axisymmetric and now only the angular momentum component parallel to the z axis is conserved, while the x and y components are not conserved**
- **The orbit follows a "rosette" pattern, but there is now motion out of the $x - y$ plane and the orbit precesses about the z axis**

This illustrates the trajectory of a star in an oblate elliptical galaxy, for example

Integrals of the motion

To solve the collisionless Boltzmann equation for stars in a galaxy - need further constraints on the position and velocity - use *integrals of the motion*

These are:

- **functions of a star's position x and velocity v that are constant along its orbit**
- **useful in potentials $\Phi(\mathbf{x})$ that are constant over time – the distribution function f is also constant along the orbit and can be written as a function of integrals of the motion**

Examples of integrals of the motion:

- ***Total energy* of a particular star in a potential is constant over time: $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}mv^2 + m\Phi(\mathbf{x})$**
- **In an axisymmetric potential (e.g. our Galaxy), the *z-component of the angular momentum*, L_z, is conserved. Therefore L_z is an integral of the motion in such a potential**
- **In a spherical potential, the *total angular momentum*, L, is constant. Therefore L is an integral of the motion in this potential, and the x, y and z components of L are each integrals of the motion**

Proof that Total Energy is an integral of the motion:

Energy per unit mass:

$$\begin{aligned} E_m &= \frac{1}{2}v^2 + \Phi(\mathbf{x}) \\ &= \frac{1}{2}\dot{\mathbf{x}}\bullet\dot{\mathbf{x}} + \Phi(\mathbf{x}) \end{aligned}$$

Differentiating w.r.t. time:

$$\begin{aligned} \frac{\mathrm{d}E_m}{\mathrm{d}t} &= \dot{\mathbf{x}}\bullet\ddot{\mathbf{x}} + \frac{\mathrm{d}\Phi(\mathbf{x})}{\mathrm{d}t} \\ &= \dot{\mathbf{x}}\bullet\ddot{\mathbf{x}} + \frac{\partial\Phi(\mathbf{x})}{\partial t} + \dot{\mathbf{x}}\bullet(\boldsymbol{\nabla}\Phi) \\ &= \dot{\mathbf{x}}\bullet\ddot{\mathbf{x}} + \dot{\mathbf{x}}\bullet(-\ddot{\mathbf{x}}) \\ &= 0 \end{aligned}$$

Proof that z-component of the angular momentum in an axisymmetric potential is an integral of the motion:

$$L_z \equiv \mathbf{L}\bullet\hat{\mathbf{e}}_z = m(\mathbf{r}\times\mathbf{v})\bullet\hat{\mathbf{e}}_z$$

Differentiating w.r.t. time:

$$\begin{aligned} \frac{\mathrm{d}L_z}{\mathrm{d}t} &= m(\dot{\mathbf{r}}\times\dot{\mathbf{r}} + \mathbf{r}\times\ddot{\mathbf{r}})\bullet\hat{\mathbf{e}}_z \\ &= m(\mathbf{r}\times\ddot{\mathbf{r}})\bullet\hat{\mathbf{e}}_z \\ &= 0 \end{aligned}$$

Proof that the total angular momentum in a spherical potential is an integral of the motion:

$$\mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$

Differentiating w.r.t. time:

$$\begin{aligned} \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= m(\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}) \\ &= m(\mathbf{r} \times \ddot{\mathbf{r}}) \\ &= 0 \end{aligned}$$

Isolating integrals and integrable systems

The CBE implies df/d$t = 0$

So if we move with a star in its orbit, f is constant locally as the star passes through phase space at that instant in time

If the system is also in a steady state (the potential is constant over time), f is constant along the star's path *at all times* - the orbits of stars map out constant values of f

An integral of the motion for a star (e.g. energy per unit mass, E_m) is constant (by definition) and

- **one value of the integral therefore defines a 5-D hypersurface in 6-D phase space**
- **the motion of a star is confined to that 5-D surface in phase space**
- **so f is constant over that hypersurface**

A different value of the isolating integral (e.g. a different value of E_m)

- **defines a different hypersurface**
- **f will be different on this surface**

So f is a function of the isolating integral i.e. $f(x, y, z, v_x, v_y, v_z) = \mathbf{fn}(I_1)$ where I_1 is an integral of the motion and I_1 here "isolates" a hypersurface

Therefore the integral of the motion is known as an *isolating integral*

An orbit is said to be *regular* (i.e. not chaotic) if it has as many isolating integrals that can define the orbit unambiguously as there are spatial dimensions. It is also said to be *integrable*

If isolating integrals exist, then any f that depends only on them will automatically satisfy the collisionless Boltzmann equation

This result is known as the *Jeans Theorem*