Chapter 2
STELLAR DYNAMICS

The Virial Theorem

The *virial theorem* is an important relation between the total kinetic energy and total gravitational potential energy for dynamically evolved systems of stars, such as some galaxies

The stellar systems must be dynamically evolved (relaxed), must be isolated (no external forces) and must be gravitationally bound

The *virial theorem* states that for any system of particles bound by an inverse-square force law, the time-averaged total kinetic energy $\langle T \rangle$ and the time-averaged total potential energy $\langle U \rangle$ are related by:

$$\mathbf{2\,\langle T\rangle + \langle U\rangle = 0}$$

for a steady equilibrium state. $\langle T \rangle$ will be a very large positive quantity and $\langle U \rangle$ a very large negative quantity

In practice, the virial theorem does not apply exactly, but does give important, approximate results for many astronomical systems

The Virial Theorem : derivation

Start with a system of N stars, where the ith star has mass m_i and position vector $\mathbf{x}_i$

Consider the moment of inertia I of this system of stars:

$$I \equiv \sum_{i=1}^{N} m_i \, \mathbf{x}_i \bullet \mathbf{x}_i = \sum_{i=1}^{N} m_i \, x_i^2$$

Differentiating with respect to time t:

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \, \mathbf{x}_i \bullet \mathbf{x}_i\right) \\
&= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \, \mathbf{x}_i \bullet \mathbf{x}_i \right) \\
&= \sum_i m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{x}_i \bullet \mathbf{x}_i \right) \qquad \textbf{assuming constant masses, } m_i \\
&= \sum_i m_i \left(\dot{\mathbf{x}}_i \bullet \mathbf{x}_i + \mathbf{x}_i \bullet \dot{\mathbf{x}}_i \right) \qquad \textbf{product rule} \\
&= 2 \sum_i m_i \, \dot{\mathbf{x}}_i \bullet \mathbf{x}_i
\end{aligned}$$

Differentiating again:

$$\begin{aligned}
\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} &= 2\,\frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i\,\dot{\mathbf{x}}_i\bullet\mathbf{x}_i \\
&= 2\sum_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(m_i\,\dot{\mathbf{x}}_i\bullet\mathbf{x}_i\Big) \\
&= 2\sum_i m_i\,\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\mathbf{x}}_i\bullet\mathbf{x}_i\Big) \\
&= 2\sum_i m_i\,\Big(\dot{\mathbf{x}}_i\bullet\dot{\mathbf{x}}_i \,+\, \ddot{\mathbf{x}}_i\bullet\mathbf{x}_i\Big) \\
&= 2\sum_i m_i\,\dot{x}_i^2 \;+\; 2\sum_i m_i\,\ddot{\mathbf{x}}_i\bullet\mathbf{x}_i
\end{aligned}$$

The kinetic energy of the ith particle is $\frac{1}{2}m_i\dot{x}_i{}^2$, so the total kinetic energy of the entire system of stars is:

$$T \;=\; \sum_i \tfrac{1}{2}\,m_i\,\dot{x}_i^2 \qquad\qquad \therefore\;\; \sum_i m_i\,\dot{x}_i^2 \;=\; 2\,T$$

Substituting:

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 4\,T \;+\; 2\sum_i m_i\,\ddot{\mathbf{x}}_i\bullet\mathbf{x}_i$$

at any time t

The average of any parameter $y(t)$ over time $t = 0$ to τ is

$$\langle y\rangle \;=\; \frac{1}{\tau}\int_0^\tau y(t)\,\mathrm{d}t$$

So we now consider the average value of $\mathrm{d}^2 I/\mathrm{d}t^2$ over a time interval $t = 0$ to τ :

$$
\begin{aligned}
\left\langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \right\rangle &= \frac{1}{\tau} \int_0^\tau \left(\; 4\,T \; + \; 2 \sum_i m_i \, \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \; \right) \, \mathrm{d}t \\
&= \frac{4}{\tau} \int_0^\tau T \, \mathrm{d}t \; + \; \frac{2}{\tau} \int_0^\tau \sum_i m_i \, \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \; \mathrm{d}t \\
&= 4 \, \langle T \rangle \; + \; 2 \sum_i \frac{m_i}{\tau} \int_0^\tau \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \; \mathrm{d}t
\end{aligned}
$$

assuming constant masses, m_i

$$
= \; 4 \, \langle T \rangle \; + \; 2 \sum_i m_i \, \langle \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \rangle
$$

When the system of stars reaches equilibrium, the moment of inertia I will be constant. Therefore :

$$
\left\langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \right\rangle = 0
$$

This is true because I will be bounded in any physical system, therefore :

$$
\lim_{\tau \to \infty} \langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \rangle = \lim_{\tau \to \infty} (\frac{1}{\tau} \int_0^\tau \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \, \mathrm{d}t) \; \to \; 0
$$

because $\mathbf{d}^2 I/\mathrm{d}t^2$ remains finite

Substituting for $\langle \mathrm{d}^2 I/\mathrm{d}t^2 \rangle = 0$:

$$4\,\langle T \rangle \;+\; 2\sum_i m_i\,\langle \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \rangle \;=\; 0$$

$$\therefore\; 2\,\langle T \rangle \;+\; \sum_i m_i\,\langle \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \rangle \;=\; 0$$

The term $\sum_i m_i \langle \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \rangle$ is related to the gravitational potential. But how?

Newton's Second Law of Motion gives for the ith star:

$$m_i\,\ddot{\mathbf{x}}_i \;=\; \sum_{\substack{j \\ j\neq i}} \mathbf{F}_{ij}$$

where $\mathbf{F}_{ij}$ is the force exerted on the ith star by the jth star

Newton's law of universal gravitation then gives:

$$m_i\,\ddot{\mathbf{x}}_i \;=\; \sum_{\substack{j \\ j\neq i}} -\frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j)$$

Taking the scalar product (dot product) with $\mathbf{x}_i$

$$m_i\,\ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;=\; \Bigg(\sum_{\substack{j \\ j\neq i}} -\frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j) \Bigg) \bullet \mathbf{x}_i$$

Then summing over all the stars in the system gives:

$$\sum_i m_i\,\ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;=\; -\sum_i \sum_{\substack{j \\ j\neq i}} \frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i$$

$$\;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i$$

Switching i and j, we have

$$\sum_j m_j \, \ddot{\mathbf{x}}_j \bullet \mathbf{x}_j \;=\; -\sum_{\substack{j,i \\ i\neq j}} \frac{G\, m_j\, m_i}{|\mathbf{x}_j - \mathbf{x}_i|^3} \, (\mathbf{x}_j - \mathbf{x}_i) \bullet \mathbf{x}_j$$

Adding Equations these equations:

$$\sum_i m_i \, \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;+\; \sum_j m_j \, \ddot{\mathbf{x}}_j \bullet \mathbf{x}_j \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \, (\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i \;-\; \sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_j\, m_i}{|\mathbf{x}_j - \mathbf{x}_i|^3} \, (\mathbf{x}_j - \mathbf{x}_i) \bullet \mathbf{x}_j$$

Therefore

$$2 \sum_i m_i \, \ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \left((\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i \;+\; (\mathbf{x}_j - \mathbf{x}_i) \bullet \mathbf{x}_j \right)$$

But

$$\begin{aligned}
(\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i \;+\; (\mathbf{x}_j - \mathbf{x}_i) \bullet \mathbf{x}_j \;&=\; (\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_i \;-\; (\mathbf{x}_i - \mathbf{x}_j) \bullet \mathbf{x}_j \\
&=\; (\mathbf{x}_i - \mathbf{x}_j) \bullet (\mathbf{x}_i - \mathbf{x}_j) \\
&=\; |\mathbf{x}_i - \mathbf{x}_j|^2
\end{aligned}$$

$$\therefore \quad 2\sum_i m_i\,\ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;=\; -\sum_{\substack{i,j\\ i\neq j}} \frac{G\,m_i\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|^3}\;|\mathbf{x}_i-\mathbf{x}_j|^2$$

$$\therefore \quad \sum_i m_i\,\ddot{\mathbf{x}}_i \bullet \mathbf{x}_i \;=\; -\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{G\,m_i\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|}$$

We now need to find the total potential energy of the system

The gravitational potential at star i due to star j is

$$\Phi_{i\,j} \;=\; -\frac{G\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|}$$

Therefore the gravitational potential at star i due to all other stars is

$$\Phi_i \;=\; \sum_{\substack{j\\ j\neq i}} \Phi_{i\,j} \;=\; \sum_{\substack{j\\ j\neq i}} -\frac{G\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|}$$

Therefore the gravitational potential energy of star i due to all the other stars is

$$U_i \;=\; m_i\,\Phi_i \;=\; -\,m_i \sum_{\substack{j\\ j\neq i}} \frac{G\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|}$$

The total potential energy of the system is therefore

$$U \;=\; \sum_i U_i \;=\; \frac{1}{2}\sum_i \left(-\,m_i \sum_{\substack{j\\ j\neq i}} \frac{G\,m_j}{|\mathbf{x}_i-\mathbf{x}_j|}\right)$$

The factor $\frac{1}{2}$ ensures that we only count each pair of stars once

Therefore

$$U \;=\; -\,\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}} \frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Substituting for the total potential energy:

$$\sum_i m_i\,\ddot{\mathbf{x}}_i\bullet\mathbf{x}_i \;=\; U$$

We need time-averaged quantities, so averaging over time $t = 0$ to τ

$$\frac{1}{\tau}\int_0^\tau \sum_i m_i\,\ddot{\mathbf{x}}_i\bullet\mathbf{x}_i\,\mathrm{d}t \;=\; \langle U\rangle$$

$$\therefore\quad \sum_i m_i\,\frac{1}{\tau}\int_0^\tau \ddot{\mathbf{x}}_i\bullet\mathbf{x}_i\,\mathrm{d}t \;=\; \langle U\rangle$$

$$\therefore\quad \sum_i m_i\,\langle\,\ddot{\mathbf{x}}_i\bullet\mathbf{x}_i\,\rangle \;=\; \langle U\rangle$$

Substituting this, we finally obtain:

$$2\,\langle T\rangle \;+\; \langle U\rangle \;=\; 0$$

This completes the derivation of the Virial Theorem

Using the Virial Theorem

The virial theorem in general is applicable when a system is in dynamical equilibrium ('relaxed')

For the specific form of the virial theorem that we have derived (the scalar version), the system must also be isolated (no external forces) and non-rotating

The theorem can be applied, for example, to:

- Elliptical galaxies
- Evolved star clusters, e.g. globular clusters
- Evolved clusters of galaxies (with the galaxies acting as the particles, not the individual stars)

Examples of places where the virial theorem cannot be used are:

- Merging galaxies
- Newly formed star clusters
- Clusters of galaxies that are still forming/still have infalling galaxies

The virial theorem can be used to estimate masses:

- Measure the observed velocity dispersion of stars
- Calculate the total kinetic energy
- Determine the total gravitational potential energy and hence the mass from the virial theorem

Example 1: deriving masses - the uniform sphere

Consider a spherical elliptical galaxy of radius R, with uniform density and consisting of N stars each of mass m having typical velocities v

From the virial theorem,

$$2\,\langle T\rangle \;+\; \langle U\rangle \;=\; 0$$

where $\langle T\rangle$ is the time-averaged total kinetic energy and $\langle U\rangle$ is the average total potential energy.

So therefore

$$T \;=\; \sum_{i=1}^{N} \frac{1}{2} m v^2 \;=\; \frac{1}{2} N m v^2$$

and averaging over time

$$\langle T\rangle \;=\; \frac{1}{2} N m v^2$$

(Note that strictly speaking we are taking the typical velocity to be the root mean square velocity i.e. the velocity dispersion)

The total gravitational potential energy of a uniform sphere of mass M and radius R (a standard result) is

$$U \;=\; -\,\frac{3}{5}\,\frac{GM^2}{R}$$

where G is the universal gravitational constant

So the time-averaged potential energy of the galaxy is

$$\langle U \rangle \;=\; -\,\frac{3}{5}\,\frac{GM^2}{R}$$

where M **is the total mass**

Substituting this into the virial theorem equation gives

$$2\left(\frac{1}{2}Nmv^2\right) \;-\; \frac{3}{5}\,\frac{GM^2}{R} \;=\; 0$$

But the total mass is $M = Nm$

$$\therefore \quad v^2 \;=\; \frac{3}{5}\,\frac{NGm}{R} \;=\; \frac{3}{5}\,\frac{GM}{R}$$

The calculation is only approximate, so we can use

$$v^2 \;\simeq\; \frac{NGm}{R} \;\simeq\; \frac{GM}{R}$$

So the mass is

$$M \;\simeq\; \frac{v^2 R}{G}$$

So an elliptical galaxy having a typical velocity $v = 350\,\mathrm{km\,s^{-1}} = 3.5\times10^5\,\mathrm{m\,s^{-1}}$**, and a radius** $R = 10$ **kpc** $= 3.1\times10^{20}$ **m, will have a mass** $M \sim 6\times10^{41}$ **kg** $\sim 3\times10^{11} M_\odot$

Example 2: the Fundamental Plane for elliptical galaxies

Using the Virial Theorem, we can derive a relationship between scale size, central surface brightness and central velocity dispersion for elliptical galaxies, similar to the Fundamental Plane (Section 1.3), assuming only that

- **the mass-to-light ratio is constant for ellipticals (all E galaxies have the same M/L regardless of their size or mass), and**
- **elliptical galaxies have the same functional form for the mass distribution, only scalable**

Let I_0 be the central surface brightness and R_0 be a scale size of a galaxy (in this case, different galaxies will have different values of I_0 and R_0). The total luminosity will be

$$L \;\propto\; I_0\, R_0^2$$

because I_0 is the light per unit projected area

Since the mass-to-light ratio is a constant for all galaxies, the mass of the galaxy is $M \;\propto\; L$

$$\therefore\;\; M \;\propto\; I_0\, R_0^2$$

From the virial theorem, if v is a typical velocity of the stars in the galaxy

$$v^2 \;\simeq\; \frac{GM}{R_0}$$

The observed velocity dispersion along the line sight, σ_0, will be related to the typical velocity v by $\sigma_0 \propto v$ (because v is a three-dimensional space velocity). So

$$\sigma_0^2 \;\propto\; \frac{M}{R_0} \qquad\qquad \therefore\;\; M \;\propto\; \sigma_0^2\, R_0$$

Equating this with $M \;\propto\; I_0\, R_0^2$ from above, we get:

$$\sigma_0^2\, R_0 \;\propto\; I_0\, R_0^2$$
$$\therefore\;\; R_0\, I_0\, \sigma_0^{-2} \;\;\simeq\;\; \textbf{constant}$$

This is close to the observed Fundamental Plane result $R_0\, I_0^{0.8}\, \sigma_0^{-1.3} \simeq$ constant

The difference is probably due to varying mass-to-light ratio - if the galaxies had different ages, their luminosities could be different

The Crossing Time, T_{cross}

The *crossing time* is a simple, but important, parameter that measures the timescale for stars to move significantly within a system of stars

It is sometimes called the *dynamical timescale*

It is defined as

$$T_{cross} \equiv \frac{R_s}{v}$$

where R_s is the size of the system and v is a typical velocity of the stars

Simple example: a stellar system of radius R (so overall size $R_s = 2R$), with N stars each of mass m; the stars are distributed roughly homogeneously, with v being a typical velocity, and the system is in dynamical equilibrium

From the virial theorem

$$v^2 \simeq \frac{NGm}{R}$$

The crossing time is then

$$T_{cross} \equiv \frac{2R}{v} \simeq \frac{2R}{\sqrt{\frac{NGm}{R}}} \simeq 2\sqrt{\frac{R^3}{NGm}}$$

But the mass density is

$$\rho = \frac{Nm}{\frac{4}{3}\pi R^3} = \frac{3Nm}{4\pi R^3}$$

$$\therefore \frac{R^3}{Nm} = \frac{3}{4\pi\rho}$$

$$\therefore T_{cross} = 2\sqrt{\frac{3}{4\pi G\rho}}$$

So approximately

$$T_{cross} \sim \frac{1}{\sqrt{G\rho}}$$

Although derived for a homogeneous sphere, this is an important result – useful for order of magnitude estimates in other situations

(Note ρ here is the mass density of the system, averaged over a volume of space, and not the density of individual stars)

Example: for an elliptical galaxy of 10^{11} stars and radius 10 kpc :

$$R \simeq 10\,\mathbf{kpc} \simeq 3.1 \times 10^{20}\,\mathbf{m}$$
$$N = 10^{11}$$
$$m \simeq 1\,M_\odot \simeq 2 \times 10^{30}\,\mathbf{kg}$$

$T_{cross} \simeq 2\sqrt{\frac{R^3}{NGm}}$ **gives** $T_{cross} \simeq 10^{15}$ **s** $\simeq 10^8$ **yr**

The Universe is 14 Gyr old. So if a galaxy is $\simeq 14$ Gyr old, there are $\simeq$ few $\times 100$ crossing times in a galaxy's lifetime so far

The Relaxation Time, T_{relax}

The relaxation time is the time taken for a star's velocity v to be changed significantly by two-body interactions

It is defined as the time needed for a change Δv^2 in v^2 to be the same as v^2 i.e. the time for

$$\Delta v^2 = v^2$$

To estimate the relaxation time we need to consider the nature of encounters between stars in some detail

Star-star encounters

Close encounters between stars can alter the stars' orbits as a result of their mutual gravitational attraction

What is the strength of this effect?

We can consider two different types of star-star encounter:

- ***strong encounters*** **– a close encounter that strongly changes a star's velocity – these are** ***very*** **rare in practice**

- ***weak encounters*** **– occur at a very large distance – they produce only very small changes in a star's velocity, but are much more common**

Strong encounters

A strong encounter between two stars is defined such that, at closest approach, the change in the potential energy is greater than or equal to the initial kinetic energy

For two stars of mass m that approach to a distance r, this implies:

$$\frac{Gm^2}{r} \geq \frac{1}{2}mv^2$$

where v is the initial velocity of one star relative to the other

$$\therefore \quad r \leq \frac{2Gm}{v^2}$$

We define a strong encounter radius

$$r_S \equiv \frac{2Gm}{v^2}$$

and a strong encounter occurs if two stars approach to within distance r_S

For an elliptical galaxy, with $v \simeq 300$ kms^{-1}. Using $m = 1M_\odot$, we find that $r_S \simeq 3 \times 10^9$ m $\simeq 0.02$ AU

For stars in the solar neighbourhood, $v \simeq 30$ kms^{-1} and $m = 1M_\odot$. This gives $r_S \simeq 3 \times 10^{11}$ m $\simeq 2$ AU. This again is very small on the scale of the Galaxy

So strong encounters are very rare

Distant weak encounters

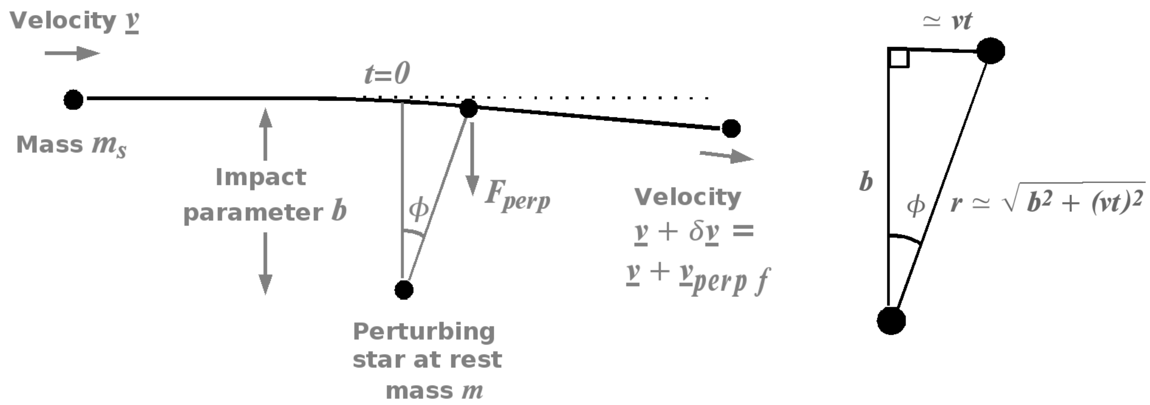
A star experiences a weak encounter if it approaches another star to within a minimum distance r_0 where

$$r_0 \;>\; r_S \;\equiv\; \frac{2Gm}{v^2}$$

v is the relative velocity before the encounter and m is the mass of the perturbing star

In weak encounters, perturbations are small but frequent, so more important cumulatively than strong encounters

Consider a star, mass m_s, approaching a perturbing star of mass m with impact parameter i.e. closest approach distance, b



Since the encounter is weak, changes in direction of motion will be small and changes in velocity will be perpendicular to the initial direction of motion

At any time t when the separation is r, the component of the gravitational force perpendicular to the direction of motion will be

$$F_{perp} \;=\; \frac{Gm_s m}{r^2}\cos\phi$$

where ϕ is the angle at the perturbing mass between the point of closest approach and the perturbed star

Let the component of velocity perpendicular to the initial direction of motion be v_{perp} and the final value $v_{perp\,f}$

Assuming the speed along the trajectory is constant, $r \simeq \sqrt{b^2 + v^2t^2}$ **at time t if $t = 0$ at the point of closest approach**

Using $\cos\phi = b/r \simeq b/\sqrt{b^2 + v^2t^2}$ **and applying $F = ma$ perpendicular to the direction of motion:**

$$\frac{\mathrm{d}v_{perp}}{\mathrm{d}t} \;=\; \frac{G\,m\,b}{(b^2 + v^2t^2)^{3/2}}$$

Integrating from time $t = -\infty$ to ∞ :

$$\Big[\,v_{perp}\,\Big]_0^{v_{perp\,f}} \;=\; G\,m\,b\int_{-\infty}^{\infty}\frac{\mathrm{d}t}{(b^2 + v^2t^2)^{3/2}}$$

Using the standard integral $\int_{-\infty}^{\infty}(1+s^2)^{-3/2}\,\mathrm{d}s = 2$ **with substitution $s = \tan x$, we obtain:**

$$v_{perp\,f} \;=\; \frac{2Gm}{bv}$$

Since the deflection is small, the change of velocity is $\delta v \equiv |\delta \mathbf{v}| = v_{perp\, f}$ so:

$$\delta v = \frac{2Gm}{bv}$$

These velocity changes will tend to cancel so the sum over all δv will remain small, but the sum of the squares δv^2 will increase with time

What happens to δv^2 during a single encounter?

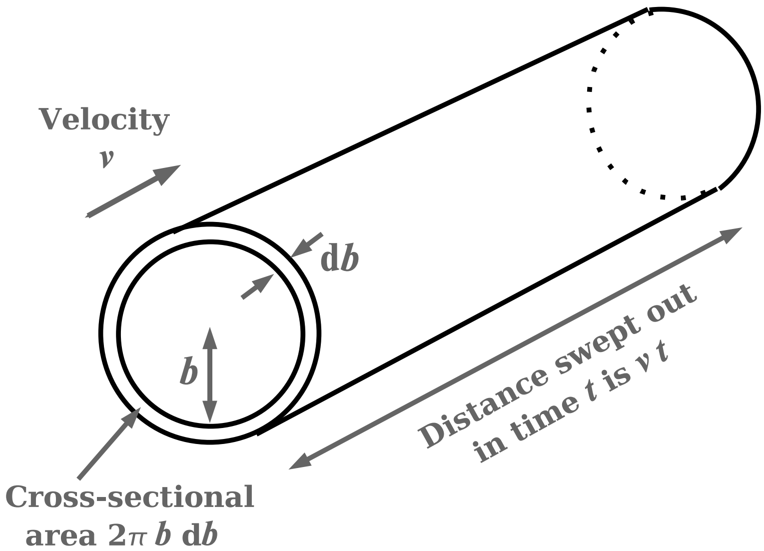
Because the change in velocity δv is perpendicular to the initial velocity v in a weak encounter, the change in v^2

$$\begin{aligned}
\delta v^2 \equiv v_f^2 - v^2 &= |\mathbf{v} + \delta\mathbf{v}|^2 - v^2 \\
&= (\mathbf{v} + \delta\mathbf{v}) \bullet (\mathbf{v} + \delta\mathbf{v}) - v^2 \\
&= \mathbf{v} \bullet \mathbf{v} + 2\mathbf{v} \bullet \delta\mathbf{v} + \delta\mathbf{v} \bullet \delta\mathbf{v} - v^2 \\
&= 2\mathbf{v} \bullet \delta\mathbf{v} + (\delta v)^2 \\
&= (\delta v)^2
\end{aligned}$$

So δv^2, the change in v^2 resulting from a single encounter is actually equal to the square of the change in v, $(\delta v)^2$, and therefore:

$$\delta v^2 = \left(\frac{2Gm}{bv}\right)^2$$

Now let's consider all weak encounters occurring in a time period t that have impact parameters in the range b to $b + \mathrm{d}b$ within a uniform spherical system of N stars and radius R



The volume swept out by impact parameters b to $b + \mathrm{d}b$ in time t is $2\,\pi\,b\,\mathrm{d}b\,v\,t$

So the number of stars encountered with impact parameters between b and $b + \mathrm{d}b$ in time t is

(volume swept out) (number density of stars)

$$= \Big(2\,\pi\,b\,\mathrm{d}b\,v\,t\Big)\,\frac{N}{\frac{4}{3}\pi R^3} \;=\; \frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2R^3}$$

The total change in v^2 caused by all encounters in time t with impact parameters in the range b to $b+\mathrm{d}b$ will be

(Δv^2 in 1 encounter) (number of encounters)

$$\Delta v^2 \quad = \quad \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right)$$

Integrating over b, the total change in a time t from all impact parameters from b_{min} to b_{max} is

$$\Delta v^2(t) \quad = \quad \int_{b_{min}}^{b_{max}} \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right)$$

$$= \quad \frac{3}{2}\left(\frac{2Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \int_{b_{min}}^{b_{max}} \frac{\mathrm{d}b}{b}$$

$$= \quad 6\left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

Crossing time revisited

How much does v^2 change in one crossing time?

In one crossing time $T_{cross} = 2R/v$, the change in v^2 is:

$$\begin{aligned}\Delta v^2(T_{cross}) &= 6 \left(\frac{Gm}{v}\right)^2 \frac{v}{R^3} \left(\frac{2R}{v}\right) N \ln\left(\frac{b_{max}}{b_{min}}\right) \\ &= 12\,N \left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{b_{max}}{b_{min}}\right)\end{aligned}$$

The maximum scale over which weak encounters will occur corresponds to the size of the system of stars. So use $b_{max} \simeq R$ and we get :

$$\Delta v^2(T_{cross}) = 12\,N \left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{R}{b_{min}}\right)$$

Relaxation time revisited

The relaxation time is defined as the time taken for $\Delta v^2 = v^2$

Substituting for Δv^2 :

$$6 \left(\frac{Gm}{v}\right)^2 \frac{v\, T_{relax}\, N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) = v^2$$

$$\therefore \quad T_{relax} = \frac{1}{6N \ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2}$$

or putting $b_{max} \simeq R$

$$T_{relax} = \frac{1}{6N \ln\left(\frac{R}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2}$$

This equations allows us to estimate the relaxation time for a system of stars, such as a galaxy or a globular cluster.

Different derivations can have slightly different numerical constants because of the different assumptions made.

What about b_{min} and b_{max}?

b_{min} is often set to the scale on which strong encounters begin to operate, so $b_{min} \simeq 1$ AU

In practice, b_{max} is the scale of the system of stars: use $b_{max} \simeq R$

Examples of relaxation time calculations

1. **For an elliptical galaxy:**

$v \simeq 300$ **kms**$^{-1} = 3.0 \times 10^5$ **ms**$^{-1}$
$N \simeq 10^{11}$ **stars**
$R \simeq 10$ **kpc** $\simeq 3.1 \times 10^{20}$ **m**
$m \simeq 1\ M_\odot \ \simeq\ 2.0 \times 10^{30}$ **kg**

Therefore, $\ln(R/b_{min}) \simeq 21$

and $T_{relax} \sim 10^{24}$ **s** $\sim 10^{17}$ **yr**

The Universe is 14×10^9 **yr old. So weak star-star encounters are of no significance for galaxies**

2. **For a large globular cluster:**

$v \simeq 10$ **kms**$^{-1} = 10^4$ **ms**$^{-1}$
$N \simeq 500\,000$ **stars**
$R \simeq 5$ **pc** $\simeq 1.6 \times 10^{17}$ **m**
$m \simeq 1\ M_\odot \ \simeq\ 2.0 \times 10^{30}$ **kg**

Therefore, $\ln(R/b_{min}) \simeq 15$

and $T_{relax} \sim 5 \times 10^{15}$ **s** $\sim 10^7$ **yr**

This is a small fraction (10^{-3}) **of the age of the Galaxy**

Two body interactions are therefore significant in globular clusters

As a result, there is a gradual redistribution of the orbits of stars in globular clusters over time

- **cluster cores collapse with more massive stars and binaries congregating in the core**

- **less massive stars can be ejected from the cluster ('evaporation')**

Ratio of the Relaxation Time to the Crossing Time

Dividing the expressions for the relaxation and crossing times:

$$\frac{T_{relax}}{T_{cross}} = \frac{\frac{1}{6N\ln\left(\frac{R}{b_{min}}\right)}\frac{(Rv)^3}{(Gm)^2}}{\frac{2R}{v}} = \frac{1}{12N\ln\left(\frac{R}{b_{min}}\right)}\frac{R^2v^4}{(Gm)^2}$$

For a uniform sphere, from the virial theorem:

$$v^2 \simeq \frac{NGm}{R}$$

Set b_{min} equal to the strong encounter radius:

$$r_S = \frac{2Gm}{v^2}$$

Therefore:

$$\frac{T_{relax}}{T_{cross}} = \frac{1}{12N\ln\left(\frac{Rv^2}{2Gm}\right)}\frac{R^2v^4}{(Gm)^2} \simeq \frac{N^2}{12N\ln(N)}$$

$$\therefore \frac{T_{relax}}{T_{cross}} \simeq \frac{N}{12\ln N}$$

For a galaxy, $N \sim 10^{11}$. Therefore $T_{relax}/T_{cross} \sim 10^9$
For a globular cluster, $N \sim 10^5$ and $T_{relax}/T_{cross} \sim 10^3$

So, again, we conclude that encounters are not significant for galaxies - they behave as *collisionless* systems

Collisional and Collisionless Systems

In chapter 1 we described dynamical systems as being either *collisional* or *collisionless* as follows:

- **collisional if interactions between individual particles substantially affect their motions**

- **collisionless if interactions between individual particles *do not* substantially affect their motions**

The relaxation time calculations showed that

- **galaxies are in general *collisionless* systems (but the region around the central nuclei of galaxies where the density of stars is very large could be collisional)**

- **Globular clusters are *collisional* over the lifetime of the Universe**

- **Gas, whether in galaxies or in the laboratory, is *collisional***

Modelling is much easier if two-body encounters can be ignored (i.e. the system is *collisionless*)

Fortunately, we can ignore star-star interactions when modelling galaxies and this makes possible the use of the *Collisionless Boltzmann Equation* (more later)

Violent relaxation

Stars in galaxies are collisionless objects. So stars in a steady state galaxy will continue in steady state orbits without perturbing each other

The average distribution of stars will therefore not change with time

However, the situation can be very different if the system is not in equilibrium

A changing gravitational potential (with time) will cause the orbits of the stars to change

But since the stars determine the overall potential, the change in their orbits will change the potential

This change in the motions of stars caused by changes in their net potential is called *violent relaxation*

Galaxies experienced violent relaxation during their formation - bringing them to their current equilibrium state

Interactions between galaxies can also cause violent relaxation

Violent relaxation happens relatively quickly ($\sim 10^8$ yr)