Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH6105 / MTH6105P: Algorithmic Graph Theory

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

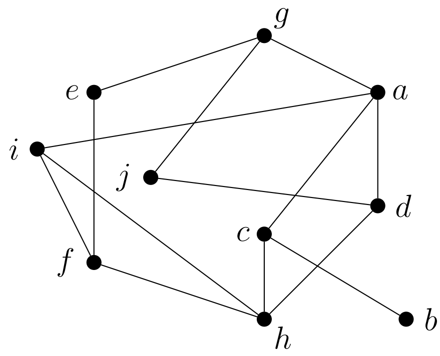
When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: F. Fischer, V. Latora

You may use any result from lecture notes and exercises without proving it, but you must state clearly which result you use.

Question 1 [24 marks]. Let G be the graph given by the following drawing.



(a) Draw the induced subgraph of G on vertex set $\{a, d, e, f, g, h\}$. [**4**]

(b) Draw a subgraph of G that is isomorphic to the graph H with $V(H) = \{r, s, t, u, v, w, x, y, z\}$ and $E(H) = \{rs, rt, ru, st, sv, tw, ux, vy, wz\}$. [**4**]

(c) Draw a spanning tree of G whose set of leaves is $\{a, b, g, j\}$, or explain why such a spanning tree does not exist. [**4**]

Call a cycle of a graph H a **Hamiltonian cycle** of H if it contains every vertex of H.

(d) Give a Hamiltonian cycle of G, or explain why such a cycle does not exist. [**4**]

Let G be an arbitrary simple graph, $n = |V(G)|$, and $m = |E(G)|$.

(e) Assume that the complement of G is connected. Show that $m \leq \frac{1}{2}n^2 - \frac{3}{2}n + 1$. [**8**]

Question 2 [24 marks]. Let D be a directed graph.

(a) Assume that D is a directed acyclic graph. Prove or disprove that there exists a vertex $v \in V(D)$ such that $d_D^+(v) = 0$. [**4**]

(b) Assume that there exists a vertex $v \in V(D)$ such that $d_D^+(v) = 0$. Prove or disprove that D must be a directed acyclic graph. [**4**]

Consider the directed network (D, w) with $V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $A(D) = \{v_1v_3, v_1v_5, v_2v_4, v_2v_6, v_3v_2, v_3v_6, v_4v_6, v_5v_2, v_5v_3\}$, and

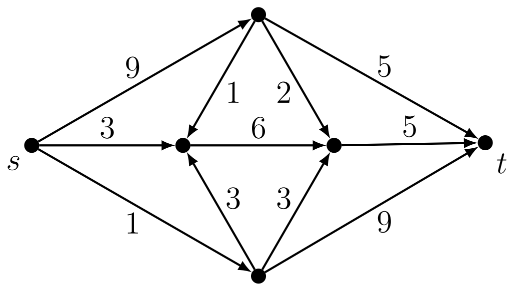
$$w(v_1v_3) = 5, \quad w(v_1v_5) = 4, \quad w(v_2v_4) = 1, \quad w(v_2v_6) = 3, \quad w(v_3v_2) = 2,$$
$$w(v_3v_6) = 5, \quad w(v_4v_6) = 1, \quad w(v_5v_2) = 7, \quad w(v_5v_3) = 3.$$

(c) Draw the network. [**4**]

(d) Show that D is a directed acyclic graph. [**4**]

(e) Use Morávek's algorithm to find a longest v_1–v_6-path of (D, w). Show your working and give the path and its length. [**8**]

Question 3 [26 marks]. Consider the directed network (D, c) given by the following drawing, where each arc $e \in A(D)$ is labeled by its capacity $c(e)$ and two vertices s and t have been identified.



(a) Use the Ford-Fulkerson algorithm to find a maximum s–t-flow of (D, c). Draw the residual network after each iteration of the algorithm, and give the size of the maximum flow. [**10**]

(b) Use a cut to show that the flow you have found is indeed a maximum s–t-flow of (D, c). [**6**]

(c) If the capacity of both of the arcs with capacity 9 was decreased to 6, how would this affect the size of a maximum flow? Justify your answer. [**4**]

Call a directed network (D, c) an s–t-**bottleneck network**, where $s, t \in V(D)$, if for every arc $uv \in A(D)$ there exists a minimum s–t-cut S of (D, c) with $u \in S$ and $v \notin S$.

(d) Give an efficient algorithm that determines, for a given directed network (D, c) with $c : A(D) \to \mathbb{N}$ and $s, t \in V(D)$, whether (D, c) is an s–t-bottleneck network. Explain briefly why the algorithm is correct and efficient. [**6**]

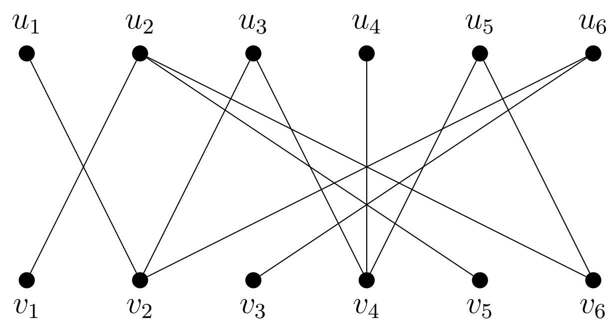
Question 4 [26 marks].

(a) Give a tree T with $|V(T)| = 7$ that has exactly two maximum matchings. Justify your answer. **[4]**

(b) Let G be an acyclic graph with $V(G) \neq \emptyset$. Show that there exists $v \in V(G)$ with $d_G(v) = 0$, or there exist $u, v \in V(G)$ with $u \neq v$ and $d_G(u) = d_G(v) = 1$. **[4]**

(c) Let T be a tree, M_1 and M_2 distinct maximum matchings of T. Show that $|M_1| + |M_2| < |V(T)|$. **[6]**

Consider the tree T given by the following drawing.



(d) Find a maximum matching of T. Show your working. **[8]**

(e) Give a set $X \subseteq \{u_1, u_2, u_3, u_4, u_5, u_6\}$ with $|X| > |N_T(X)|$, or explain why such a set does not exist. **[4]**

End of Paper.