Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH5114: Linear Programming and Games

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: J. Ward, F. Fischer

Question 1 [30 marks].

(a) Consider the following linear program:

$$\begin{array}{lrcl} \text{minimise} & x_1 + 2x_2 - 6x_3 & & \\ \text{subject to} & 4x_1 - \ \ 8x_2 + \ \ x_3 & = & 10, \\ & 3x_1 - \ \ 9x_2 \qquad & \leq & 20, \\ & 2x_1 + 10x_2 + 2x_3 & \geq & 30, \\ & x_1 \qquad\qquad & \leq & 40, \\ & x_1, x_2 & \geq & 0, \\ & x_3 & & \text{unrestricted} \end{array} \tag{1}$$

(i) Convert program (1) to standard inequality form, and give the value of the constraint matrix A, the vector $\mathbf{b}$ (which gives the right-hand side of the constraints), and the vector $\mathbf{c}$ (which gives the coefficients for the objective function). [**6**]

(ii) Give the dual of program (1). [**6**]

(b) Consider a linear program in standard inequality form:

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{2}$$

Let $t > 0$ be some constant and suppose we modify the program by multiplying the right-hand side of each constraint by t:

$$\begin{array}{ll} \text{maximise} & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \leq t\mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{3}$$

(i) Show that $\mathbf{x}$ is a feasible solution of program (2) if and only if $t\mathbf{x}$ is a feasible solution of program (3). [**6**]

(ii) Show that if program (2) is unbounded then the program (3) is unbounded. [**6**]

(iii) Show that if $t\mathbf{x}$ is an optimal solution of program (3) then $\mathbf{x}$ is an optimal solution of program (2).
Hint: it may be easier to prove the contrapositive: that if $\mathbf{x}$ is **not** an optimal solution for (2) then $t\mathbf{x}$ is **not** an optimal solution to (3). [**6**]

Question 2 [25 marks].

(a) Solve the following linear program using the simplex algorithm. Give each tableau produced by the algorithm, using the same format used during lectures.

Also state the optimal solution and its objective value or explain why an optimal solution does not exist.

$$\begin{aligned} \text{maximise} \quad & -x_1 + 2x_2 - x_3 \\ \text{subject to} \quad & -2x_1 + 4x_2 + 2x_3 \leq 8, \\ & -2x_1 + 2x_2 + 4x_3 \leq 2, \\ & x_1 + x_2 - 2x_3 \leq 6, \\ & x_1, x_2, x_3 \geq 0p \end{aligned}$$

[10]

(b) Suppose we want to solve the following linear program, in which α is a fixed constant, by using the 2-phase simplex algorithm.

$$\begin{aligned} \text{maximise} \quad & x_1 + x_2 + x_3 \\ \text{subject to} \quad & \alpha x_1 + 3x_2 + 4x_3 \leq -3, \\ & x_1 + x_2 + 4x_3 \leq 2, \\ & 2\alpha x_1 + x_2 + 3x_3 = 4, \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{4}$$

(i) Give the starting phase 1 tableau for this problem. Make sure your tableau is in a valid form, and is in the same format used during lectures. **[4]**

(ii) Give a value of α for which the linear program (4) is infeasible or explain why no such value exists. Justify your answer using your starting tableau and your knowledge of the 2-phase simplex algorithm. **[5]**

(iii) Give a value of α for which an artificial variable will be selected as the entering variable in the first pivot of the algorithm. Justify your answer using your starting tableau and your knowledge of the 2-phase simplex algorithm. **[6]**

Question 3 [21 marks]. Consider the following linear program in which $\beta \in \mathbb{R}$ is a fixed constant:

$$\begin{aligned} \text{maximise} \quad & \beta x_1 + 4x_2 + 2x_3 \\ \text{subject to} \quad & x_1 - 2x_2 + 3x_3 \leq 6, \\ & -2x_1 + 2x_2 + 3x_3 \leq 1, \\ & x_1 + 2x_2 + 3x_3 \leq 4, \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{5}$$

(a) Convert the program (5) to standard equation form. **[3]**

(b) Show that this program has an extreme point solution with $x_1 = 1$, $x_2 = \frac{3}{2}$ and $x_3 = 0$. Give the values for the slack variables at this solution, as well. **[4]**

(c) Derive the complementary slackness conditions that a feasible solution $\mathbf{y}$ to the dual of (5) must satisfy when both $x_1 = 1, x_2 = \frac{3}{2}, x_3 = 0$ and $\mathbf{y}$ are optimal solutions. Explain how you obtained these conditions. **[6]**

(d) Prove that (5) has $x_1 = 1, x_2 = \frac{3}{2}, x_3 = 0$ as an optimal solution if and only if $-4 \leq \beta \leq 2$. **[8]**

Question 4 [24 marks].

(a) Consider a 2-player zero-sum game with the following payoff matrix. The matrix given, as usual, from the perspective of the row player.

	c_1	c_2	c_3	c_4
r_1	1	2	1	3
r_2	−1	0	−2	0
r_3	1	3	1	4
r_4	−2	0	−1	0

(i) Give the security levels associated with each of the strategies for both players and list all pure Nash equilibria for the game. If there are no pure Nash equlibria for the game, explain how you determined this. **[4]**

(ii) Consider a mixed strategy $(\mathbf{x}, \mathbf{y})$ in which $\mathbf{x}^{\mathsf{T}} = (1/2, 0, 1/2, 0)$ and $\mathbf{y}^{\mathsf{T}} = (1/3, 0, 2/3, 0)$. Compute the security levels associated with $\mathbf{x}$ and $\mathbf{y}$. Show your work. **[4]**

(iii) Is $(\mathbf{x}, \mathbf{y})$ a Nash equilibrium? Justify your answer. **[2]**

(b) Consider the general 2-player game with the following payoff matrix, in which the row player's payoffs are listed first in each cell:

	c_1	c_2	c_3
r_1	$(-2, 2)$	$(-3, 3)$	$(-4, 4)$
r_2	$(0, 1)$	$(1, -1)$	$(5, -5)$
r_3	$(1, -1)$	$(2, -2)$	$(4, 0)$

(i) Is this game a zero-sum game? Justify your answer. **[2]**

(ii) Show that this game does not have a pure Nash equilibrium. **[4]**

(iii) Suppose that we change the payoff to **both** players in the outcome (r_3, c_3) to be $a \in \mathbb{R}$ so that the entry $(4, 0)$ in the matrix becomes (a, a). Determine the set of all possible of values a for which (r_3, c_3) will be a pure Nash equilibrium of the resulting game. Justify your answer. **[4]**

(iv) Are there any other values of a for which the resulting game has some pure Nash equilibrium? Justify your answer. **[4]**

End of Paper.