Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5114: Linear Programming and Games

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: J. Ward, F. Rinçon

Throughout the exam, you may use (without proof) any numbered Definitions, Theorems, Corollaries, or Claims from the lecture notes to justify your answers.

When doing so, you **must give the relevant number** for any such results that you use—for example,"By definition of an optimal solution (Definition 1.7)..." or "From Theorem 4.3...".

Question 1 [18 marks]. A coffee company operates coffee roasters at 2 different facilities in the city. They have been contracted to supply 4 different restaurants with coffee. Suppose that each restaurant requires 100 kg of coffee, and that Roasters 1 and 2 have 180 kg and 220 kg, respectively, of coffee available for shipment each week.

(a) Suppose that the cost in pounds of shipping each kg of coffee between the roasters and restaurants is as follows:

	Restaurant 1	Restaurant 2	Restaurant 3	Restaurant 4
Roaster 1	1.10	1.30	2.50	0.90
Roaster 2	1.50	1.75	1.20	1.25

Give a linear program to find the cheapest way for the company to fulfil its weekly shipping contract. State what each of the variables and constraints in your program represent. You do not need to solve this program. [**8**]

(b) The coffee company wants to know how to improve its shipping costs by increasing the amount of coffee available at one of the roasters. Suppose you had an optimal solution for your linear program for the previous question. Explain how you could determine how a small increase in the amount of coffee available at one of the roasters would affect the cost of the optimal solution **without** solving the program again. [**4**]

(c) Suppose now that the company is not concerned with shipping costs but instead wants to purchase some standard packaging for its shipments. It wants to send each shipment from a roaster to a restaurant in a single box. Suppose that these boxes are identical and each can hold at most some total weight w. The company wants to know the smallest w required so that it can carry out its shipments in this way. Give a linear program to find this capacity. You do not need to solve this program. [**6**]

Question 2 [18 marks]. Consider an arbitrary linear program in standard inequality form, with n variables and m constraints:

$$\begin{aligned}
\text{maximise} \quad & c_1x_1 + c_2x_2 + \cdots c_nx_n \\
\text{subject to} \quad & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2 \\
& \qquad \vdots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{aligned} \tag{A}$$

Recall that we can transform this into the following equivalent standard equation form program with $n+m$ variables and m constraints:

$$\begin{aligned}
\text{maximise} \quad & c_1x_1 + c_2x_2 + \cdots c_nx_n \\
\text{subject to} \quad & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n + s_1 = b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n + s_2 = b_2 \\
& \qquad \vdots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n + s_m = b_m \\
& x_1, x_2, \ldots, x_n \ge 0 \\
& s_1, s_2, \ldots, s_m \ge 0
\end{aligned} \tag{B}$$

(a) Show that program (B) has an extreme point solution with $x_1 = x_2 = \ldots = x_n = 0$ if and only if $b_i \ge 0$ for all $i = 1, 2, \ldots, m$. **[8]**

(b) Show that for any extreme point solution of program (B), the corresponding solution of program (A) must have at least n tight constraints or restrictions. **[10]**

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Question 3 [20 marks].

(a) Explain the purpose of artificial variables and slack variables in the context of the 2-phase simplex algorithm. Specifically, for both kinds of variables explain why are they introduced, what each one of them measures, and what we can conclude when they are set to zero. **[6]**

(b) Consider the following program, in which a, b, and c are the last 3 digits of your student ID number (for example, if you student ID is 170123456, then you should substitute 4 for a, 5 for b, and 6 for c in the program):

$$\begin{aligned} \text{maximise} \quad & ax_1 + 3x_2 + x_3 \\ \text{subject to} \quad & -bx_1 - x_2 + 2x_3 \geq 6 \\ & -x_2 + x_3 = 1 \\ & -cx_1 + 2x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(i) Give the initial starting tableau for the 2-phase simplex algorithm for this program. **[4]**

(ii) Carry out the 2-phase simplex algorithm on this problem. You should give an optimal solution to the program or say why an optimal solution does not exist. Show your work and explain how you determined if the program is feasible or not at the end of the first phase. **[10]**

Question 4 [20 marks]. Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & -x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 \leq 2 \\ & 2x_1 + x_2 + x_3 \leq 2 \\ & x_1 + 5x_2 + x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) Give the dual of this program. **[4]**

(b) Determine whether $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/4$ is an optimal solution to this program. If it is an optimal solution, give an optimal solution $\mathbf{y}$ to the dual that together with $\mathbf{x}$ satisfies the complementary slackness conditions for the program. If it is not an optimal solution, show that no such $\mathbf{y}$ can exist. In either case, you should explicitly derive the relevant complementary slackness conditions and show whether or not they hold. **[12]**

(c) Suppose we convert this program to standard equation form. Find the values for the slack variables when x_1, x_2, x_3 are set as in the previous question. Is the resulting solution a basic feasible solution? Justify your answer. **[4]**

Question 5 [24 marks].

(a) In the following question, let $\beta \in \mathbb{R}$ be a fixed constant. Suppose a zero-sum 2-player game has the following payoff matrix, given from the perspective of the row player:

	1	2
1	β	6
2	-6	0

(i) Suppose that $\beta = 0$. Give the security levels for each of the row and column players strategies. List all pure Nash equilibria for this game or explain why the game does not have a pure Nash equilibrium. **[4]**

(ii) For what range of possible values for β is $(1, 2)$ a pure Nash equilibrium for this game? Justify your answer. **[6]**

(iii) For what range of possible values for β does this game have a general Nash equilibrium? Justify your answer. **[4]**

(b) Consider the following 2-player game. Rosemary and Colin each select a number n from the set $\{1, 2, 3\}$. If they choose the same number, neither player wins anything. Otherwise, if the sum of their numbers is at least 5, both of them win £1. Finally, if their numbers do not match and do not sum to at least 5, then the player who selected the largest number n wins £n and the other player loses £n.

(i) Give the payoff matrix for this game (as usual, suppose that Rosemary is the row player and give her payoff first in each cell). **[4]**

(ii) Is this a zero sum game? Justify your answer. **[2]**

(iii) List all pure Nash equilibria for this game. **[4]**

End of Paper.