Main Examination period 2021 – May/June – Semester B
Online Alternative Assessments

MTH5113: Introduction to Differential Geometry

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

You have **24 hours** to complete and submit this assessment. When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about 2 hours to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: A. Shao, B. Noohi

Question 1 [17 marks]. Consider the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + y^2 = 16\},$$

and consider the following parametrisation of C:

$$\gamma : (-2, 2) \to C, \qquad \gamma(t) = \left(t,\ \sqrt{16 - 4t^2}\right).$$

(a) Sketch the image of γ. [**5**]

(b) Find the unit tangents and the unit normals to C at the point $\left(\sqrt{2},\ 2\sqrt{2}\right)$. **Draw and label these on your sketch from part (a).** [**8**]

(c) Can the parametrisation γ be used to directly compute the tangent line to C at $(0, -4)$? Briefly explain why or why not. [**4**]

Question 2 [23 marks]. Consider the surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < x < 2,\ 4y^2 + 4(z+1)^2 = x^2\},$$

and consider the following parametrisation of S:

$$\sigma : (0, 1) \times \mathbb{R} \to S, \qquad \sigma(u, v) = (2u,\ u\cos v,\ u \sin v - 1).$$

(a) Sketch the image of σ. Moreover, on your sketch, indicate (i) one path obtained by holding v constant and varying u, and (ii) one path obtained by holding u constant and varying v. [**8**]

(b) Find the tangent plane to S at the point $\left(1, \frac{1}{2}, -1\right)$. [**6**]

(c) Compute the surface integral

$$\iint_S \mathbf{H} \cdot \mathbf{dA},$$

where S has the outward-facing orientation, and where $\mathbf{H}$ is the vector field

$$\mathbf{H}(x, y, z) = (x^2,\ -z-1,\ y)_{(x,y,z)}, \qquad (x, y, z) \in \mathbb{R}^3.$$ [**9**]

(Hint: The image of σ is all of S.)

Question 3 [19 marks].

(a) Find the minimum and maximum values of the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x - y,$$

subject to the constraint

$$x^4 + y^4 = 32.$$

Also, at which points are these minimum and maximum values achieved? [**15**]

(b) Dr Mistake wishes to find the maximum and minimum values of $a(x, y) = x$, subject to the constraint $b(x, y) = x^2 - y^2 = 1$. For this, Dr Mistake applies the method of Lagrange multipliers to solve the system

$$\nabla a(x, y) = \lambda \nabla b(x, y), \qquad b(x, y) = 1,$$

and correctly obtains the solutions $(x, y, \lambda) = \pm\left(1, 0, \frac{1}{2}\right)$ to this system. However, Dr Mistake then incorrectly concludes that the maximum and minimum values of $a(x, y)$ are ± 1 and are achieved at $(x, y) = \pm(1, 0)$. What did Dr Mistake do wrong? Explain Dr Mistake's mistake. [**4**]

 Continue to next page

Question 4 [20 marks].

(a) Show that the following set is a curve:

$$M = \{(x, y) \in \mathbb{R}^2 \mid x^2 - 3xy + 2y^2 = 1\}.$$

Briefly explain your reasoning. [**5**]

(b) Compute the arc length of the curve

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid 0 < z < 1,\ x = 2e^z,\ y = e^{2z}\}.$$ [**6**]

(c) Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. Dr Mistake wishes to compute the curl of the divergence of $\mathbf{F}$ (i.e. $\nabla \times (\nabla \cdot \mathbf{F})$). Why is Dr Mistake not allowed to do this? [**4**]

(d) Let C denote the unit circle about the origin,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\},$$

with the **anticlockwise** orientation, and let $\mathbf{W}$ be the vector field on $\mathbb{R}^2$ given by

$$\mathbf{W}(x, y) = (ax + by, cx + dy)_{(x,y)},$$

where $a, b, c, d \in \mathbb{R}$ are fixed constants. Apply **Green's theorem** to compute

$$\int_C \mathbf{W} \cdot ds,$$

State your answer in terms of the constants a, b, c, d. [**5**]

Question 5 [21 marks].

(a) Is the following parametric surface regular:

$$\rho : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \rho(u, v) = (u \cos v,\ u \sin v,\ u^4)?$$

Justify your answer. **[5]**

(b) Find the unit normals at the point $\left(\sqrt{2}, -\sqrt{2}, \sqrt{2}\right)$ to the hyperboloid

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid 4z^2 = x^2 + y^2 + 4\}.$$ **[6]**

(c) Let Q be the quarter-sphere,

$$Q = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x > 0,\ y > 0\}.$$

Describe the boundary of this surface Q, as one or more curves in $\mathbb{R}^3$. (You should express these curves as sets, or you can draw them for partial credit.) **[5]**

(d) Recall, from discussions in the lectures or lecture notes, that one can apply Green's theorem in order to express the area of an open region $D \subseteq \mathbb{R}^2$ in terms of integrals over the boundary of D.

Describe how one can similarly apply the divergence theorem to express the volume of an open region $V \subseteq \mathbb{R}^3$ in terms of an integral over the boundary of V. Here, you can suppose that the boundary of V is a single surface S. **[5]**

End of Paper.