Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH4115 / MTH4215: Vectors and Matrices

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the rough paper provided.

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Exam papers must not be removed from the examination room.

Examiners: A. Saha, S. Jana

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Each of the multiple choice questions below has exactly one correct answer among the choices provided. You must use an HB pencil to enter your choices in the machine-readable multiple choice answer sheet provided. Please ensure that you enter your ID number on the answer sheet and mark your answers in a manner that is sufficiently dark for the sheet to scan correctly. If you make a mistake, please use an eraser.

Question 1 [4 marks]. Let $\mathbf{u}$ and $\mathbf{v}$ be free vectors. Which of the following choices is **not** equal to the zero vector? [**4**]

(A) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$

(B) $\mathbf{u} \times \mathbf{u}$

(C) $\mathbf{v} - \mathbf{v}$

(D) $\mathbf{v} \times \mathbf{0}$

(E) $(\mathbf{u} + \mathbf{v}) + (-\mathbf{u}) - \mathbf{v}$

Question 2 [4 marks]. Let A, B be points in 3-space with position vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$. Let C be the midpoint of the line segment AB. What is $|\overrightarrow{AC}|$ equal to? [**4**]

(A) $\frac{5}{2}$

(B) 5

(C) $\sqrt{17}$

(D) $\frac{\sqrt{17}}{2}$

(E) $\sqrt{68}$

Question 3 [4 marks]. Let A, B, C be points in 3-space with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Let P be the point on the line segment AB such that $|\overrightarrow{AP}| = \frac{1}{3}|\overrightarrow{AB}|$. What is the position vector of the midpoint of the line segment PC? [**4**]

(A) $\frac{\mathbf{b}+\mathbf{c}}{2}$

(B) $\frac{\mathbf{a}+\mathbf{b}}{2}$

(C) $\frac{2\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

(D) $\frac{2\mathbf{a}+\mathbf{b}+3\mathbf{c}}{6}$

(E) $\frac{\mathbf{a}+2\mathbf{b}+3\mathbf{c}}{6}$

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Question 4 [4 marks]. Let $A = (1, 2, 2)$ and $B = (2, 3, 4)$. Which of the following points lies on the line through A and B? [**4**]

(A) $(-1, 0, -2)$

(B) $(3, 5, 6)$

(C) $(0, 0, 0)$

(D) $(2, 3, 5)$

(E) $(-2, -3, -4)$

Question 5 [4 marks]. Let O denote the origin, and let A, B be points in 3-space with position vectors $\mathbf{a}$, $\mathbf{b}$ respectively. Let M be the plane passing through O, A and B, and suppose that C is a point with position vector $\mathbf{c}$ such that C lies on M. Which of the following expressions must be true? [**4**]

(A) $(\mathbf{b} \times \mathbf{c}) = \mathbf{a}$

(B) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$

(C) $(\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 0$

(D) $\mathbf{a} \cdot \mathbf{b} = 0$

(E) $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

Question 6 [4 marks]. Let Π be the plane with equation $x + y + 2z = 1$ and let Q be the point with position vector $\mathbf{q} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. What is the distance between the point Q and the plane Π? [**4**]

(A) $\sqrt{5}$

(B) $\frac{5}{\sqrt{6}}$

(C) $\frac{5}{6}$

(D) 2

(E) $\frac{3}{\sqrt{2}}$

Question 7 [4 marks]. Let ℓ_1 be the line with vector equation $\mathbf{r} = \mathbf{p} + \lambda\mathbf{u}$ and let ℓ_2 be the line with vector equation $\mathbf{r} = \mathbf{q} + \mu\mathbf{v}$. Let A be a point on ℓ_1, and let B be a point on ℓ_2, so that $|\overrightarrow{AB}|$ is minimised. Let $\mathbf{w}$ be the vector represented by $\overrightarrow{AB}$. Which of the following expressions must be true? [**4**]

(A) $\mathbf{w} = \mathbf{q} - \mathbf{p}$

(B) $\mathbf{w} = \mathbf{u} - \mathbf{v}$

(C) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(D) $\mathbf{w} = \mathbf{u}$

(E) $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = 0$

Question 8 [4 marks]. Let A, B, C, D be points in 3-space with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ respectively. Suppose that $ACBD$ is a parallelogram. Which of the following expressions must be true? [**4**]

(A) $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$

(B) $\mathbf{a} - \mathbf{d} = \mathbf{c} - \mathbf{b}$

(C) $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$

(D) $\mathbf{a} - \mathbf{c} = \mathbf{b} - \mathbf{d}$

(E) $\mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$

Question 9 [4 marks]. Let $\mathbf{u}$ and $\mathbf{v}$ be unit vectors such that $|\mathbf{u} \times \mathbf{v}| = \frac{3}{5}$. Which of the following is a possible value for $\mathbf{u} \cdot \mathbf{v}$? [**4**]

(A) 0

(B) $\frac{2}{5}$

(C) - $\frac{4}{5}$

(D) 1

(E) $\frac{3}{5}$

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Question 10 [4 marks]. Let ℓ be the line with equations $x = y = z$, and let Q be the point $(0, 1, 2)$. What is the distance between the point Q and the line ℓ? [**4**]

(A) $\sqrt{2}$

(B) $\sqrt{3}$

(C) 2

(D) $\sqrt{5}$

(E) 0

Question 11 [4 marks]. Let ℓ be the line with vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. What is the Cartesian equation for ℓ? [**4**]

(A) $x - 1 = y + 1 = z - 2$

(B) $x = 1$

(C) $y + 1 = z - 2$, $x = 1$

(D) $x = 1$, $y = -1$

(E) $y = -1$, $z = 2$

Question 12 [4 marks]. Let ℓ_1 be the line with vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and ℓ_2 be the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$. What is the distance between ℓ_1 and ℓ_2? [**4**]

(A) 2

(B) 1

(C) 0

(D) $\sqrt{2}$

(E) $\sqrt{3}$

Question 13 [4 marks]. Let Π be the plane through $(1, 2, 2)$ and orthogonal to $\mathbf{j}$. Let ℓ be the line $x = y = 2z$. Which of the following statements is true? [**4**]

(A) The intersection of Π and ℓ is the empty set.

(B) The intersection of Π and ℓ is a point.

(C) The intersection of Π and ℓ is a line.

(D) The intersection of Π and ℓ is a set of three distinct points.

(E) The intersection of Π and ℓ is a plane.

Question 14 [4 marks].
If x, y, z, and w satisfy the system of equations below

$$\begin{aligned} x - 3y + z &= 4 \\ 3x - y + z &= 2 \\ x - y &= 3, \end{aligned}$$

then what is the value of $x - y + z$? [**4**]

(A) $4/5$

(B) 1

(C) 0

(D) -3

(E) $-5/3$

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Question 15 [4 marks]. Let $A := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}$ and let $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. What is $(A^4 - A)\mathbf{v}$ equal to? [**Hint**: First compute $A\mathbf{v}$.] [**4**]

(A) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

(B) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(C) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(D) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(E) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Question 16 [4 marks]. Let

$$X := \begin{pmatrix} 3^{1/2} & 2^{1/2} \\ 2^{1/2} & 3^{1/2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3^{1/2} & -2^{1/2} \\ -2^{1/2} & 3^{1/2} \end{pmatrix}.$$

What is $\det(X^5)$ equal to? [**4**]

(A) $2^3 3^2$

(B) $2^{-15/2}$

(C) 3^5

(D) $2^5 3^5$

(E) $2^{7/2} 3^{-9/2}$

Question 17 [4 marks]. Consider the system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -x_1 + 4x_3 &= -1 \\ -2x_2 + 3x_4 &= 0 \\ x_1 &= 10. \end{aligned}$$

Which of the following statements is true? [**4**]

(A) The system is consistent and has a unique solution

(B) The system is consistent and has infinitely many solutions

(C) The system is inconsistent

(D) The system is underdetermined

(E) The system is overdetermined

Question 18 [4 marks]. Consider the system of linear equations:

$$\begin{aligned} 2x_1 - x_2 + 4x_3 &= 9 \\ 2x_2 + 5x_3 &= 21 \\ -3x_1 + x_2 - 3x_3 &= -13 \\ 3x_2 + x_3 &= 0. \end{aligned}$$

Bring the augmented matrix to the reduced row echelon form and select the correct statement among the following alternatives. [**4**]

(A) x_1 is the leading variable and x_2, x_3 are the free variables

(B) x_1, x_2 are the leading variables and x_3 is the free variable

(C) x_1, x_2, x_3 are the leading variables and there is no free variable

(D) x_1, x_3 are the leading variables and x_2 is the free variable

(E) x_2 is the leading variable and x_1, x_3 are the free variables

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Question 19 [4 marks]. Consider the matrices

$$A := \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}.$$

Which of the following statements is true? [**4**]

(A) Both A and B are in the reduced row echelon form

(B) A is not in the reduced row echelon form but B is

(C) A is in row echelon form but B is not

(D) A is in the reduced row echelon form but B is not

(E) Neither A nor B is in row echelon form

Question 20 [4 marks]. Consider the matrices

$$X := \begin{pmatrix} 2 & 3 & 4 \\ 3 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & 6 & 5 \end{pmatrix}, \quad Y := \begin{pmatrix} 2 & 6 & -9 & 0 \\ 1 & 6 & 5 & -2 \end{pmatrix}.$$

Which of the following statements is true? [**4**]

(A) X^2 exists but Y^2 does not

(B) XY exists but YX does not

(C) Both XY and YX exist

(D) Neither XY nor YX exists

(E) YX exists but XY does not

Question 21 [4 marks]. What does the determinant of the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 9 \\ 12 & -2 & 12 & 0 & 4 \\ -13 & 16 & -13 & -5 & -8 \\ -6 & 1 & -6 & 0 & -2 \\ -21 & 6 & -21 & 5 & 19 \end{pmatrix}$$

equal to? [**4**]

(A) 298

(B) -14

(C) -73

(D) 0

(E) 11

Question 22 [4 marks]. Let A be an $n \times n$ matrix. Which of the following statements is **always** true? [**4**]

(A) $A - A^\top$ is symmetric

(B) $A^\top A$ is invertible

(C) If A is lower triangular then $A^\top A$ is lower triangular

(D) If A is upper triangular and $A = A^\top$ then A is diagonal

(E) If A is both strictly upper and strictly lower triangular then A is the identity matrix

Question 23 [4 marks]. Let A be a 3×3 square matrix that can be written as a product of elementary matrices. Which of the following statements is **never** true? [**4**]

(A) A is invertible

(B) The equation $A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ has infinitely many solutions

(C) The system $A\mathbf{x} = \mathbf{0}$ only has trivial solution

(D) $\det(A) \neq 0$

(E) A is row equivalent to the identity matrix

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Question 24 [4 marks]. Which of the following conditions ensures that the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 2 & b \\ 0 & 0 & c & d & e \\ 0 & 0 & 0 & f & 0 \end{pmatrix}$$

is in the reduced row echelon form? [4]

(A) $c = 1, d = 1, f = 1$, and b, e are arbitrary

(B) $c = 0, d = 1, f = 0$, and b, e are arbitrary

(C) $c = 1, d = 0, f = 1$, and b, e are arbitrary

(D) $c = 1, f = 0$, and b, d, e are arbitrary

(E) $c = 0, d = 0, f = 1$, and b, e are arbitrary

Question 25 [4 marks]. Consider the matrix and the column vector

$$A := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

What is the length of the vector $A^{-1}\mathbf{b}$? [4]

(A) $\sqrt{3}$

(B) 2

(C) 1

(D) 3

(E) $\sqrt{2}$

End of Paper.

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