QUESTION 1

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Let x, y, z be positive integers, A, B, C be finite sets of real numbers, and $f : \mathbb{N} \to \mathbb{N}$. Classify each of the following expressions as being: a set, a number, a statement or meaningless.

$x - y$

- set
- number
- statement
- meaningless

$A \setminus B$

- set
- number
- statement
- meaningless

f is injective

- set
- number
- statement
- meaningless

$x \cup y \cup z$

- set
- number
- statement
- meaningless

$f(|A| + |B| - |A \cup B|)$

- set
- number
- statement
- meaningless

$f(f(f(x)))$

- set
- number
- statement
- meaningless

The codomain of f

- set
- number
- statement
- meaningless

$(A \cup B \cup \{x, y, z\}) \setminus \{f(x), f(y), f(z)\}$

- set
- number
- statement
- meaningless

QUESTION 2

Not yet answered Marked out of 20.0 Flag question

Let A and B be finite sets with $|A| = 8, |B| = 7, |A \cap B| = 2$. For a set X we write $\mathcal{P}(X)$ for the power set of X.

Calculate the following:

$\binom{|A|}{3} =$ ____

$|A \cup B| =$ ____

The number of non-empty subsets of B is ____

$|\{S \in \mathcal{P}(A) : |S| = 3 \text{ and } S \cap B = \emptyset\}| =$ ____

$|\{S \in \mathcal{P}(B) : |S| > 2\}| =$ ____

QUESTION 3

Not yet answered Marked out of 4.0 Flag question

You want to use induction to prove that a statement $P(n)$ holds for all $n \in \mathbb{N}$. Select from the list below, all valid ways of doing this.

Select one or more:

- ☐ $P(1)$ holds, and $P(k)$ implies $P(k+1)$ for all $k \in \mathbb{N}$
- ☐ $P(1)$ holds, and 'not $P(n)$' implies 'not $P(n+1)$' for all $n \in \mathbb{N}$
- ☐ $P(5)$ holds, and $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$
- ☐ $P(1)$ and $P(2)$ hold, and $P(n)$ implies $P(n+2)$ for all $n \in \mathbb{N}$
- ☐ $P(1)$ holds, and $P(n)$ implies $P(n+2)$ for all $n \in \mathbb{N}$
- ☐ $P(1)$ holds, and $P(n+1)$ implies $P(n)$ for all $n \in \mathbb{N}$
- ☐ $P(1)$ holds, and $P(n)$ implies $P(n+1)$ for all $n \in \mathbb{N}$
- ☐ $P(1)$ holds, and 'not $P(n+1)$' implies 'not $P(n)$' for all $n \in \mathbb{N}$

QUESTION 4

Not yet answered Marked out of 8.0 Flag question

Let $(a_k)_{k\in\mathbb{N}}$ be the sequence defined by $a_1 = 1, a_{k+1} = 3a_k - 1$ for all $k \geq 1$.

Prove that $2^k \leq a_k \leq 3^k$ for all $k \geq 5$.

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QUESTION 5

Not yet answered Marked out of 8.0 Flag question

Let the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(n, m) = (n + m, n - m)$. Decide whether f is injective and whether f is surjective, proving your answer carefully.

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QUESTION 6

Not yet answered Marked out of 12.0 Flag question

Suppose that $P(n)$, $Q(n)$ and $R(n)$ are statements about the integer n.

Let $S(n)$ be the statement: "If $P(n)$ is true then $Q(n)$ and $R(n)$ are both true."

(a) The contrapositive of $S(n)$ is the statement

- If $Q(n)$ and $R(n)$ are both true then $P(n)$ is false.
- If $Q(n)$ and $R(n)$ are both true then $P(n)$ is true.
- If at least one of $Q(n)$ and $R(n)$ is false then $P(n)$ is false.
- If $Q(n)$ and $R(n)$ are both false then $P(n)$ is false.
- If at least one of $Q(n)$ and $R(n)$ is false then $P(n)$ is true.
- If $P(n)$ is false then at least one of $Q(n)$ and $R(n)$ is false.
- If $Q(n)$ and $R(n)$ are both false then $P(n)$ is true.
- None of the other answers.

(b) The statement: "If $Q(n)$ and $R(n)$ are both true then $P(n)$ is true." is

- The converse of $S(n)$.
- The contrapositive of $S(n)$.
- The converse of the contrapositive of $S(n)$.
- None of the other answers.

(c) You want to prove that $S(n)$ is true for all $n \in \mathbb{N}$ via the contrapositive. Your proof should start by assuming that $n \in \mathbb{N}$ and:

- $P(n)$ is true.
- $P(n)$ is false.
- $Q(n)$ and $R(n)$ are both false.
- At least one of $Q(n)$ and $R(n)$ is false.
- $Q(n)$ and $R(n)$ are both true.
- At least one of $Q(n)$ and $R(n)$ is true.
- None of the other answers.

Your proof should finish by deducing that:

- $P(n)$ is true.
- $P(n)$ is false.
- $Q(n)$ and $R(n)$ are both false.
- At least one of $Q(n)$ and $R(n)$ is false.
- $Q(n)$ and $R(n)$ are both true.
- At least one of $Q(n)$ and $R(n)$ is true.
- $Q(n)$ is false.
- A contradiction (or false statement).
- None of the other answers.

QUESTION 8

Not yet answered Marked out of 12.0 Flag question

The following passage concerns a number system which we did not study in the module. Read it carefully and answer the questions that follow. All the information to do this is contained in the passage; you do not need to look at any external sources.

A real number is said to be algebraic if is the root of a polynomial with integer coefficients. For example $\sqrt{3}$ is algebraic because it is a root of $x^2 - 3$. It is easy to see that every rational number is algebraic but that not every algebraic number is rational.

In general it is rather difficult to prove that any particular irrational number is not algebraic. For instance, π is not algebraic but the proof of this (due to Lindemann in 1882) is long and complicated. Perhaps suprisingly, Lindemann's proof uses tools of integration and limits from calculus. It has also been proved that e is not algebraic, but it is unknown whether or not $e + \pi$ is algebraic.

Despite this difficulty in proving that any particular number is not algebraic, it is not too hard to show that non-algebraic numbers exist. Indeed it can be shown that the set of algebraic real numbers is countably infinite and so is in some sense much 'smaller' than the set of real numbers.

The set of algebraic real numbers satisfies some nice properties. For example it is closed under addition in the sense that the sum of two algebraic real numbers is also algebraic. This and some similar closure properties mean that the set of real algebraic numbers satisfy the definition of a mathematical structure called a field.

(a) The best title for the passage is:

- ○ Applications of Integration
- ○ Numbers
- ○ Properties of π
- ○ Roots of polynomials
- ○ A number system between $\mathbb{Q}$ and $\mathbb{R}$

(b) Which of the following polynomials shows that $(1 + \sqrt{5})$ is algebraic:

- ○ $x^2 - x - 1$
- ○ $x^2 + x + 5$
- ○ $x - (1 + \sqrt{5})$
- ○ $x^2 - 5$
- ○ $x^2 - 2x - 4$
- ○ $x^2 - 6$
- ○ $x^5 + x$

(c) To show that the number $x \in \mathbb{R}$ is not algebraic you could:

- ○ Suppose that there is a polynomial with integer coefficients and x as a root and deduce a contradiction.
- ○ Find a polynomial with integer coefficients which has x as a root.
- ○ Find a polynomial with integer coefficients which does not have x as a root.
- ○ Suppose that no polynomial with integer coefficients has x as a root and deduce a true statement.
- ○ Show that for every $y \in \mathbb{R}$ with $y \neq x$ there is a polynomial with integer coefficients which has y as a root.

QUESTION 9

Not yet answered Marked out of 3.0 Flag question

(d) Select which of the following statements relating to the material from the passage above are true:

Select one or more:

- ☐ The passage gives you enough information to fill in the details of the proof that π is not algebraic.
- ☐ For every $n \in \mathbb{N}$ the number $\sqrt{n}$ is algebraic
- ☐ There is a bijection between $\mathbb{Q}$ and the set of algebraic real numbers.
- ☐ The passage contains a precise definition of what the set of algebraic real numbers being a field means.
- ☐ The number $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is algebraic.
- ☐ There is a bijection between $\mathbb{R}$ and the set of algebraic real numbers.

QUESTION 10

Not yet answered Marked out of 5.0 Flag question

Prove carefully both the assertions made in the third sentence of the passage.

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