Queen Mary University of London

Main Examination period 2019

MTH6151: Partial Differential Equations

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: Dr. Juan A. Valiente Kroon, Dr. Shabnam Beheshti

Throughout we only consider partial differential equations in two independent variables (x, y) or (x, t).

Question 1. [20 marks]

(a) Explain, in few words, how the **method of characteristics** to solve a first order linear partial differential equation works. **[4]**

(b) Determine whether the following partial differential equations are linear or non-linear. Also, say whether they are homogeneous or inhomogeneous:

(i) $U_x + \tan x U_{yy} - U = \cos y$, **[2]**

(ii) $5UU_{tt} - U^2 U_x = 0$. **[2]**

(c) Using the method of characteristics, or otherwise, solve the equation

$$U_x - 2U_t = 0$$

subject to the condition

$$U(0, t) = \cos t.$$ **[5]**

(d) Find the general solution to the equation

$$U_t + xU_x = \sin t.$$ **[7]**

Question 2. [12 marks]

(a) Classify, according to type (hyperbolic, elliptic, parabolic), the equations:

(i) $2U_{xx} - 4U_{xy} - 6U_{yy} + U_x = 0.$ **[2]**

(ii) $U_{xx} + 2U_{xy} + 17U_{yy} = 0.$ **[2]**

(b) Suppose $f(x)$ is a differentiable function.

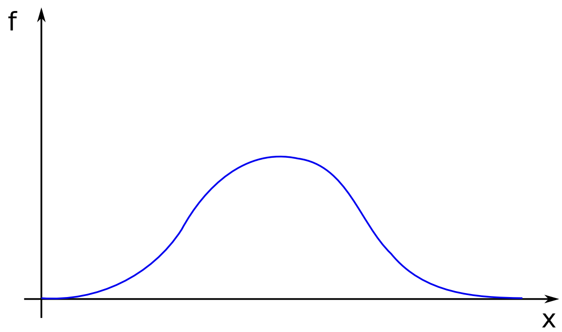
(i) Show that

$$U(x,t) = f(x+ct)$$

solves the partial differential equation

$$U_t - cU_x = 0.$$ **[3]**

(ii) If f has the form



describe the qualitative behaviour of the solution $U(x,t)$ given in (i). **[3]**

(iii) What happens with the solution if $U(x,0) = 0$? **[2]**

Question 3. [20 marks]

(a) D'Alembert's formula is given by

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.$$

Provide a brief discussion of the meaning of the two terms in the right-hand side of the above formula. **[4]**

(b) Let $U(x,t)$ denote a solution to the wave equation

$$U_{tt} - c^2 U_{xx} = 0.$$

Show that

$$V(x,t) \equiv U(\alpha x, \alpha t)$$

is also a solution to the wave equation for any constant α. **[6]**

(c) Find the solution to the problem

$$\begin{aligned} &U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}, \\ &U(x,0) = \frac{1}{1+x^2}, \\ &U_t(x,0) = 0. \end{aligned}$$

Provide a sketch of the solution for different times. **[6]**

(d) What is the main difference between the wave equation and the heat equation in terms of the speed of propagation of information? **[4]**

Question 4. [28 marks]
Throughout this question, consider the following problem for the Laplace equation on a rectangle:

$$\begin{aligned} &U_{xx} + U_{yy} = 0, \qquad (x,y) \in \Omega = \{0 < x < a,\ 0 < y < b\}, \\ &U(x,0) = 0 \qquad U(x,b) = f(x), \\ &U(0,y) = 0, \qquad U(a,y) = 0. \end{aligned}$$

(a) Following the method of separation of variables consider solutions of the form

$$U(x,y) = X(x)Y(y)$$

where X and Y are functions of a single argument. Show that X and Y satisfy the ordinary differential equations

$$\begin{aligned} X'' &= kX, \\ Y'' &= -kY, \end{aligned}$$

for some constant k. Moreover, show that

$$X(0) = X(a) = 0, \qquad Y(0) = 0.$$ **[6]**

(b) Show that the constant k obtained in (a) must be negative if $X(x)$ is not identically 0 for $x \in [0,a]$. **[6]**

(c) Find the general solution to the ordinary differential equations in (a). **[4]**

(d) Use the conditions $X(0) = X(a) = 0$ to determine the value of k and show that the non-zero solutions X obtained in (c) must be of the form

$$X(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n = 1, 2, 3, \ldots.$$

Moreover, show that if $Y(0) = 0$ then

$$Y(y) = \sinh\left(\frac{n\pi y}{a}\right).$$ **[4]**

(e) Use the **Principle of Superposition** to find the general solution to the Laplace equation on the rectangle Ω with the prescribed boundary conditions. **[4]**

(f) Assuming that the general solution to the problem can be written as

$$U(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

where a_n are constants, find the particular solution corresponding to the initial data

$$U(x,b) = \sin\left(\frac{5\pi x}{a}\right) + 2\sin\left(\frac{6\pi x}{a}\right).$$ **[4]**

Question 5. [20 marks]

(a) Briefly explain the significance of the **Fourier-Poisson formula** in the study of the heat equation. **[4]**

(b) Show that

$$U(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\varkappa t}} e^{-s^2} ds,$$

is a solution to the heat equation

$$U_t = \varkappa U_{xx}.$$

Find the value of $\lim_{t \to 0^+} U(x,t)$ if $x > 0$. **[6]**

(c) Explain what is the **Maximum Principle** for the heat equation. **[4]**

(d) Consider the solution

$$U(x,t) = 1 - x^2 - 2\varkappa t$$

of the heat equation

$$U_t = \varkappa U_{xx}.$$

Find the location of its maxima and minima in the rectangle

$$\{0 \leq x \leq 1,\ 0 \leq t \leq T\}.$$ **[6]**

End of Paper – An appendix of 1 page follows.

The Laplacian in polar coordinates

The expression for the Laplacian for a function U on $\mathbb{R}^2$ in standard spherical coordinates (r, θ) is given by

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Orthogonality properties of the sine function

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L/2 & \text{for} \quad n = m \\ 0 & \text{for} \quad n \neq m \end{cases}.$$

Gaussian integral

$$\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

D'Alembert's formula

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$

where

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

The Fourier-Poisson formula

$$U(x,t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}} f(y)dy.$$

End of Appendix.