Queen Mary University of London

Main Examination period 2018

MTH6934: Topics in Probability and Stochastic Processes

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: Dr David Ellis

Question 1. [25 marks]

(a) Let $\{N(t) : t \geq 0\}$ be a renewal process with interarrival times X_i (for $i \in \mathbb{N}$), which are independent, identically distributed random variables with mean $\mu = \mathbb{E}[X_1]$, where $0 < \mu < \infty$. Let $m(t) = \mathbb{E}[N(t)]$ for each $t \geq 0$. What does the Elementary Renewal Theorem say about $\lim_{t\to\infty} \frac{N(t)}{t}$? What does it say about $\lim_{t\to\infty} \frac{m(t)}{t}$? [**3**]

(b) State the Renewal Reward theorem. [**4**]

Suppose that a smoke alarm holds one battery at a time. When a battery fails, it is replaced immediately. It is replaced with a brand-A battery with probability 2/5 and with a brand-B battery with probability 3/5. The lifetime of each brand-A battery (measured in years) has the exponential distribution with parameter $\frac{1}{3}$ years^{-1}, i.e. it has pdf

$$f_A(x) = \begin{cases} \frac{1}{3}e^{-x/3} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The lifetime of a brand-B battery (measured in years) has pdf

$$f_B(x) = \begin{cases} \frac{3}{4}x(2-x) & \text{if } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The lifetimes of all bulbs are independent of one another.

(c) Write down, or calculate, the expected lifetime of a brand-A battery. [**2**]

(d) Find the expected lifetime of a brand-B battery. [**3**]

(e) In the long run, what is the average number of battery-replacements per year? [**4**]

(f) Suppose that a brand-A battery costs £3 and a brand-B battery costs £2. In the long run, what is the average cost per year of replacing batteries in the smoke alarm? [**5**]

(g) Suppose now that you have a choice of either using *only* brand-A batteries, or *only* using brand-B batteries. Which strategy would be more cost-effective in the long run? Justify your answer. [**4**]

Question 2. [30 marks]

(a) State the *Superposition Lemma* and the *Thinning Lemma* for Poisson processes. [**5**]

Two archers, Alice and Bob, play a game where they both shoot arrows at a target. They both start shooting at the same time. Alice scores hits on the target according to a Poisson process of rate 2 per minute. Bob scores hits on the target according to a Poisson process of rate 1 per minute. Your answers to the following questions should be expressed in terms of powers of e, where necessary, but they should be simplified as much as possible in all other ways.

(b) What is the probability that Alice has scored just one hit after 3 minutes? [**3**]

(c) What is the probability that Alice has scored just one hit after 3 minutes and just four hits (in total) after 5 minutes? [**4**]

(d) What is the probability that the total number of hits (by both Alice and Bob) is 2, after 3 minutes? [**4**]

(e) Suppose that each of Alice's hits is a bull's-eye with probability 1/8, independently of all other hits. What is the expected time until Alice scores her first bull's-eye? [**5**]

(f) Suppose now that the game lasts for 2 minutes and that at the end of the game, the score is 4 hits to Alice and 2 hits to Bob. Conditional on this information,

   (i) What is the probability that Alice has scored just one hit after one minute of the game (i.e., halfway through the game)? [**4**]

   (ii) What is the probability that Bob is ahead of Alice after one minute of the game? [**5**]

Question 3. **[20 marks]** A robot has just three speeds: 5 mph, 10 mph and 30 mph. When it changes speed to 30 mph, it remains at that speed on average for 2 hours before changing speed to either 10 mph (with probability 3/4) or to 5 mph (with probability 1/4). When it changes speed to 10 mph, it remains at that speed on average for 4 hours before changing speed to either 5 mph (with probability 1/2) or to 30 mph (with probability 1/2). When it changes speed to 5 mph, it remains at that speed on average for 1 hour before changing speed to either 10 mph (with probability 3/4) or to 30 mph (with probability 1/4).

(a) What extra assumption do we need to make in order to model the changing speed of the robot as a semi-Markov process? [**3**]

(b) Assume that the changing speed of the robot forms a semi-Markov process. Label the states as 1, 2 and 3 (in increasing order of speed). Write down the transition matrix P of the associated discrete-time Markov chain with the same state-space and transition-probabilities. [**3**]

(c) Find an equilibrium distribution $\pi = (\pi_1, \pi_2, \pi_3)$ for the associated discrete-time Markov chain, and show that it is the unique equilibrium distribution, by solving the equations

$$\pi\ P = \pi, \quad \sum_{i=1}^{3} \pi_i = 1,$$

and showing that the solution is unique. [**7**]

(d) Estimate the proportion of time the robot spends at each of the three speeds, over a very long time-period. [**3**]

(e) Assume that the robot travels in a straight line without reversing direction. Estimate the average speed of the robot over a very long time-period. [**4**]

Question 4. [25 marks]

(a) Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with generator matrix G, finite state space S, and time-t transition matrices $\{P(t) : t \geq 0\}$. Let $\pi = (\pi_i)_{i \in S}$ be a probability distribution on S. Define (in terms of the generator matrix G, or otherwise) what it means for π to be an *equilibrium distribution* for the continuous-time Markov chain $\{X(t) : t \geq 0\}$. Define (in terms of the matrices $\{P(t) : t \geq 0\}$, or otherwise) what it means for π to be a *limiting distribution* for the Markov chain. [**4**]

A small post office has two clerks. The customers in the post office who are waiting to be served, join a single queue. If there are at most two customers in the post office, the time until the arrival of the next customer is exponentially distributed with mean 5 minutes, but if there are three customers in the post office, no new customers enter. Each clerk can only serve one customer at a time, and the time taken to serve each customer is exponentially distributed with mean 10 minutes. After being served, a customer leaves.

(b) Let $Y(t)$ denote the total number of customers in the post office, t minutes after it has opened. Find the generator matrix G for the continuous-time Markov chain $\{Y(t) : t \geq 0\}$, indicating how the rows and columns of G are indexed by the states. [**6**]

(c) Find an equilibrium distribution for the continuous-time Markov chain in part (b), and show that it is the unique equilibrium distribution. [**7**]

(d) Explain why any state of this Markov chain communicates with any other state. [**2**]

(e) State, with justification, the limiting distribution of this Markov chain. (You may appeal to any standard theorem or fact from the course, without proving it.) [**3**]

(f) Estimate the proportion of time for which there are three customers in the post office, in the long run. [**3**]

End of Paper.