Queen Mary University of London

Main Examination period 2017

MTH751U / MTH751P / MTHM751
Processes on Networks

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: G. Bianconi & V. Latora

Question 1. [40 marks]

Link percolation of uncorrelated networks.
Consider an uncorrelated random network with degree distribution $P(k)$ where the *links* are randomly damaged.
Consider the following recursive algorithm to predict which nodes are in the giant component of the network:

- *A node is in the giant connected component if at least one of its links is not damaged and reaches a node in the giant component.*
- *A node reached by following a link is in the giant component, if there is at least one of its remaining links that is not damaged and that reaches a node in the giant component.*

Let S be the probability that a node is in the giant component.
Let S' be the probability a link is not initially damaged and reaches a node that is in the giant component.
Let p denote the probability that a link is not initially damaged.
The brackets $\langle \ldots \rangle$ indicate the average over the degree distribution $P(k)$.

a) Show that S' satisfies the equation

$$S' = p\left[1 - \sum_{k=0}^{\infty} \frac{kP(k)}{\langle k \rangle}(1-S')^{k-1}\right].$$

[9]

b) Show that S satisfies the equation

$$S = \left[1 - \sum_{k=0}^{\infty} P(k)(1-S')^{k}\right].$$

[8]

c) Show that, in order to have a giant component in the network, i.e. $S > 0$, we must have

$$p\frac{\langle k(k-1) \rangle}{\langle k \rangle} > 1.$$

[10]

d) Calculate the percolation threshold p_c for a network with degree distribution $P(k) = \delta_{k,5}$ where $\delta_{x,y} = 1$ if $x = y$ and otherwise $\delta_{x,y} = 0$. **[3]**

e) Consider a scale-free network with degree distribution $P(k) = ck^{-\gamma}$, with $\gamma = 3$ and $k \in [m, \sqrt{N}]$. Calculate $\langle k \rangle$ and $\langle k^2 \rangle$ in the continuous approximation. Using these results evaluate percolation threshold p_c of the netwrk in the limit $N \to \infty$. **[10]**

Question 2. [30 marks]

Robustness of uncorrelated networks to targeted attack of the high degree nodes.
Consider an uncorrelated random network with degree distribution $P(k)$.

- We initially damage a fraction f of nodes with highest degree.
- We indicate with $k_c(f)$ the highest degree of the nodes that are not initially damaged.
- We indicate by S the probability that a node is in the giant component.
- We indicate by S' the probability that a link reaches a non damaged node of degree $k \leq k_c(f)$ that is in the giant component.
- The brackets $\langle \ldots \rangle$ indicate the average over the degree distribution $P(k)$.

a) Express f as a function of k_c and of the degree distribution $P(k)$. **[5]**

b) Given an infinite scale-free network with degree distribution $P(k) = Ck^{-\gamma}$ with $\gamma = 3$ and $k \geq 1$, using the continuous approximation, show that the cutoff k_c resulting from the initial attack of 1% of the nodes with highest degree is $k_c(f) = 10$. [**10**]

c) Show that S' satisfies the equation

$$S' = \sum_k \frac{kP(k)}{\langle k \rangle} \theta(k_c(f) - k) \left[1 - (1 - S')^{k-1}\right],$$

where $\theta(x) = 1$ if $x \geq 0$ otherwise $\theta(x) = 0$. [**5**]

d) Show that S satisfies the equation

$$S = \sum_k P(k) \theta(k_c(f) - k) \left[1 - (1 - S')^k\right].$$

[**5**]

e) Show that in order to have a giant component in the network, i.e. $S > 0$ we must have

$$\frac{\langle k^2 \theta(k_c(f) - k) \rangle - \langle k \theta(k_c(f) - k) \rangle}{\langle k \rangle} > 1.$$

[**5**]

Question 3. [30 marks]

The SIS model: the annealed approximation

In the annealed approximation for the SIS model the dynamical equation for the probability ρ_k that a node of degree k is infected is given by

$$\frac{d\rho_k}{dt} = -\rho_k + \lambda k(1-\rho_k)\Theta(\lambda),$$

where

$$\Theta(\lambda) = \sum_k \frac{k}{\langle k \rangle} P(k)\rho_k,$$

and λ indicates the infectivity of the epidemics.

a) Assuming self-consistently that the value of $\Theta(\lambda)$ is known, find the stationary state of Eq. (1). **[5]**

b) Close the self-consistent argument showing that $\Theta(\lambda)$ satisfies

$$\Theta(\lambda) = \lambda \sum_k \frac{k}{\langle k \rangle} P(k) \frac{k\Theta(\lambda)}{1+\lambda k \Theta(\lambda)}.$$

[5]

c) Show that the Eq. (1) has a non trivial solution $\Theta(\lambda) > 0$ if and only if

$$\lambda > \lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle}.$$

[10]

d) What is the epidemic threshold λ_c for power-law networks with power-law exponent $\gamma \in (2, 3]$? **[4]**

e) Consider infinite scale-free network with power-law exponent $\gamma = 2.5$ (network A) and an infinite regular network of constant degree $k = 5$ (network B).
According to the annealed approximation, is it possible that the SIS epidemics spreads in network A for values of the infectivity λ for which it does not spread in network B? (*Justify your answer*). **[3]**

f) Consider an infinite regular network of constant degree $k = 5$ (network B) and an infinite regular network of constant degree $k = 3$ (network C).
According to the annealed approximation, is it possible that the SIS epidemics with given infectivty λ spreads in network C but does not spread in network B ? (*Justify your answer*). **[3]**

End of Paper.