Queen Mary
University of London

MTH750U: Graphs and Networks

Duration: 3 hours

Date and time: 7 June 2016, 10:00–13:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): V. Latora

Question 1 (34 marks).
Consider the two graphs G_1 and G_2, respectively described by the two adjacency matrices:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Are the two graphs directed or undirected? How is this related to the properties of their adjacency matrix? Find the order and size (number of links/arcs) of the two graphs, and draw them. **[9]**

(b) Are the two graphs connected? Find the degree distributions of the two graphs. **[8]**

(c) Find the matrices of distances between nodes for the two graphs. **[6]**

(d) How are node closeness and node betweenness centrality defined? Evaluate the closeness and betweenness centrality of node 3 in graph G_2. Notice that the normalised versions of centrality are required. **[6]**

(e) Give the definitions of node clustering coefficients and clustering coefficient of a graph. Evaluate the node clustering coefficient of each of the nodes in graph G_1. What is the clustering coefficient of graph G_1? **[5]**

Question 2 (33 marks).
Given the ensemble of Erdős-Renyí (ER) random graphs $G^{\text{ER}}_{N,p}$ with $N = 1000$ nodes, and where each couple of nodes is connected with a probability p, consider the two cases $p = 0.0005$ and $p = 0.01$.

(a) Evaluate the expected number of edges $\overline{K}$ in a graph, (i.e. the average number of edges in a graph, where the average is performed over the graphs in the ensemble), in the two cases $p = 0.0005$ and $p = 0.01$. **[6]**

(b) Find an expression for the probability of finding a graph with K edges in the ensemble with $p = 0.01$? What is the probability of finding an empty graph (a graph with $K = 0$ edges) in the ensemble? **[7]**

(c) Find an expression for the node degree distribution p_k of the graphs in the two ensembles. Approximate the degree distribution by a Poisson distribution, and find the values of averages, $\langle k \rangle$, and standard deviations, σ_k, of node degrees in the two cases (averages are performed over graph nodes). **[8]**

(d) State the Molloy-Reed criterion. Do the networks in the two ensembles have a giant connected component? **[7]**

(e) Find a general expression for $\overline{n}_{\Delta}$, the average number of triangles in $G^{\text{ER}}_{N,p}$ as a function of N and p. What is the average number of triangles in the two ensembles considered in this question? **[5]**

Question 3 (33 marks).
Consider the following model to grow graphs. Given three positive integers $N \gg 1$, n_0 and m (with $m \leqslant n_0$), the graph grows, starting at time $t = 0$ with a complete graph with n_0 nodes, and by iteratively repeating at time $t = 1, 2, 3, \ldots, N - n_0$, the two steps:

(1) A new node, labeled by the index n, being $n = n_0 + t$, is added to the graph. The node arrives together with m edges.

(2) The m edges link the new node to m different nodes already present in the system with equal probability, i.e. the probability $\Pi_{n \to i}$ that a new link connects the new node n to node i (with $i = 1, 2, \cdots, n - 1$) is:

$$\Pi_{n \to i} = \frac{1}{n_{t-1}}$$

where $n_t = n_0 + t$ is the number of nodes in the graph at time t.

(a) Find an expression for the number of links, l_t, as a function of time t. What is the average node degree $\langle k \rangle$ when $N \to \infty$? **[8]**

(b) Write down the rate equations of the model, i.e. the equations for $\overline{n}_{k,t}$, where $\overline{n}_{k,t}$ denotes the average number of nodes with degree k ($k \geqslant m$) present in the graph at time t. The average, as usual, is performed over infinite realizations of the growth process with the same parameters N, n_0 and m. **[8]**

(c) Solve the rate equations to find the stationary degree distribution p_k. Notice that p_k is the limit of $p_{k,t} = \overline{n}_{k,t}/n_t$ when $t \to \infty$. **[6]**

(d) Find the expression for the stationary degree distribution p_k in the case of large m. Does this model produce scale-free networks? **[4]**

(e) Write down and solve the differential equation governing the time evolution of the average degree $\overline{k_i}(t)$ of a node i in the so-called mean-field approximation. **[7]**

End of Paper.