Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH745U: Further Topics in Algebra (Rings and Modules)

Duration: 3 hours

Date and time: 5th May 2015, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray

Important information: please read carefully.
You may assume the definition of *ring*, and that all rings have a 1, except if the contrary be stated. Unless otherwise specified, R-modules are *unital left* R-modules in this paper.
We use $\mathrm{M}_n(R)$ to denote the set of $n \times n$ matrices over R, and $\mathrm{T}_n(R)$ to denote the set of upper triangular matrices in $\mathrm{M}_n(R)$, each endowed with the usual matrix multiplication. If you appeal to bookwork at any point, please indicate which results you appeal to.

Question 1 (28 marks).

(a) Define the notion of a *module* over a ring R. **[3]**

(b) Define the notion of a *homomorphism* of R-modules. **[3]**

(c) Define the notion of the *kernel* of a homomorphism of R-modules, and prove that it is a submodule of the domain. **[5]**

(d) Suppose I is an (2-sided) ideal of a (unital) ring R. Write down a homomorphism ϕ of R-modules such that $I = \ker\phi$. Justify your answer. **[5]**

(e) Let F be a field, let $R = \mathrm{T}_3(F)$, and let $M = F^3$, with the usual action of matrices on **column** vectors. Classify, with proof, all R-submodules of M. **[12]**

Question 2 (10 marks). Let R be a ring, and let M and N be R-modules.

(a) State the First Isomorphism Theorem for R-modules. **[3]**

(b) Prove that if K and L are submodules of M then $(K+L)/L \cong K/(K \cap L)$. **[7]**

Question 3 (12 marks). In this question we do not insist that a ring be unital.

(a) Let R (with operations $+$ and $\cdot$) be a ring. Define its *opposite ring* R^{op}. **[3]**

(b) Let $\mathbb{H} = \{a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} : a,b,c,d \in \mathbb{R}\}$ be 'the' ring of quaternions, and let n be a positive integer. Give an explicit isomorphism between $\mathrm{M}_n(\mathbb{H})$ and $\mathrm{M}_n(\mathbb{H})^{\mathrm{op}}$. [You need not prove the map you write down is an isomorphism.] **[4]**

(c) Give an example of a ring R such that R and R^{op} are not isomorphic as rings. Justify your answer. [Hint: let R be a subring of $\mathrm{M}_n(F)$ for some suitable integer n and field F. It may be easier to find a non-unital example.] **[5]**

Question 4 (26 marks). In this question R is a (unital) ring.

(a) Let $r \in R$. Define $\mathrm{C}_R(r)$, the *centraliser* of r in R, and prove that it is a (unital) subring of R. [**6**]

(b) What does it mean to say that R is a *division ring*? [**2**]

(c) Prove that if R is a division ring, then so is $\mathrm{C}_R(r)$ for any $r \in R$. [**3**]

(d) Give an example of a non-commutative division ring R, and a non-commutative (unital) subring S thereof, such that S is not a division ring. Justify carefully that S is not commutative, and that S is not a division ring. [You do not have to show that R is a division ring, or that S is a subring of R.] [**7**]

(e) Let M be an R-module. Define the set $\mathrm{End}_R(M)$. [**2**]

(f) Define the notion of a *simple* R-module. [**2**]

(g) Prove that if M is a simple R-module then $\mathrm{End}_R(M)$ is a division ring. [You may assume that $\mathrm{End}_R(M)$ is a ring.] [**4**]

Question 5 (24 marks). In this question R is a ring, M is an R-module, and N is a submodule of M.

(a) Define what is means for M to be *noetherian*. [**3**]

(b) Define what is means for M to be *artinian*. [**3**]

(c) Prove that M is noetherian if and only if every submodule of M is finitely generated. [**10**]

(d) Give examples (one for each part) of **nonzero** $\mathbb{Z}$-modules M such that:

(i) M is both noetherian and artinian; [**2**]

(ii) M is noetherian but not artinian; [**2**]

(iii) M is neither noetherian nor artinian; [**2**]

(iv) M is artinian but not noetherian. [**2**]

[You need not provide any justification for your examples.]

End of Paper.