Queen Mary
University of London

M. Sc. Examination by course unit 2014

MTH745P: Further Topics in Algebra (Fields and Galois Theory)

Duration: 3 hours

Date and time: 20th May 2014, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): J. N. Bray

Question 1 (30 marks) Let $K \geqslant F$ be a field extension.

(a) What does it mean for $\alpha \in K$ to be *algebraic* over F, and what does it mean for α to be *transcendental* over F? **[3]**

(b) Suppose α is algebraic over F. Define the *minimal polynomial* of α over F. **[3]**

(c) Suppose α is algebraic over F, and has minimal polynomial $m(X)(\in F[X])$ over F. Prove that $m(X)$ is irreducible over F. State a relationship between the degree of $m(X)$ and the degree of the field extension $F(\alpha) : F$. Show that if $f(X) \in F[X]$ satisfies $f(\alpha) = 0$ then $m(X) \mid f(X)$ (in $F[X]$). **[9]**

(d) State and prove the Tower Law for (finite) field extensions. **[9]**

(e) Deduce that if α and β (both in K) are algebraic over F then so are $\alpha + \beta$ and $\alpha\beta$. **[3]**

(f) Prove that $\sqrt[3]{\pi}$ is algebraic over $\mathbb{Q}(\pi)$ but transcendental over $\mathbb{Q}$. [You may assume Lindemann's theorem that π is transcendental over $\mathbb{Q}$.] **[3]**

Question 2 (20 marks)

(a) Let $K \geqslant F$ (or $K : F$) be an algebraic field extension. Define what it means for this field extension to be:

(i) normal; **[3]**

(ii) separable. **[3]**

(b) Give examples, with **brief** justifications where necessary, of field extensions that are:

(i) separable and normal; **[2]**

(ii) separable and not normal; **[4]**

(iii) inseparable and normal. **[4]**

(c) Let $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{6})$ and $L = \mathbb{Q}(\sqrt[4]{6})$. Determine the normality or not of each of the extensions $K : F$, $L : K$ and $L : F$, whose degrees are 2, 2 and 4 respectively. **[4]**

Question 3 (5 marks)

(a) Write down without proof all subfields of $\mathbb{F}_{4096}$, indicating clearly the containments between them. [Recall that $4096 = 2^{12}$.] **[3]**

(b) Use this information to calculate the number of irreducible polynomials of degree 12 over $\mathbb{F}_2$. **[2]**

Question 4 (30 marks) Let $f(X) = X^3 - 4X + 2$ be a polynomial over $\mathbb{Q}$, and let α, β and γ be its three complex roots. Let $K := \mathbb{Q}(\alpha, \beta, \gamma)$ be the splitting field of f over $\mathbb{Q}$.

(a) Factorise $X^3 - 4X + 2$ into irreducible factors over K, in terms of α, β and γ. **[1]**

(b) Write down the value of $\alpha + \beta + \gamma$. **[1]**

(c) Determine (with justification) the number of real roots of f. **[3]**

(d) Prove that f is irreducible (over $\mathbb{Q}$). **[2]**

(e) Determine the degree of the extension $K : \mathbb{Q}$. [Hint: You may assume that $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha) = \pm\sqrt{148}$ (the sign depends on the order of α, β, γ).] **[5]**

(f) Calculate the Galois group $\mathrm{Gal}(K : \mathbb{Q})$. **[5]**

(g) List all subfields of K and the containments between them. [You do not need to determine the normality or separability of the field extensions involved.] **[8]**

(h) Factorise $X^3 - 4X + 2$ into irreducible factors over $\mathbb{Q}(\alpha)$. The coëfficients of your factors should be polynomials in α and not involve β or γ. **[5]**

Question 5 (15 marks) Let F be a field of characteristic 0, and let $n \in \mathbb{Z}^+$. We construct a tower $F \leqslant K \leqslant L$ of fields as follows. Define K to be the splitting field over F for the polynomial $X^n - 1$, and let $\zeta \in K$ be a primitive nth root of 1 (so that $\zeta^m \neq 1$ for $0 < m < n$). Now pick $0 \neq \alpha \in K$, and let L be the splitting field over K for the polynomial $X^n - \alpha$. Let $\beta \in L$ be a root of $X^n - \alpha$.

(a) Prove that $K = F(\zeta)$. **[3]**

(b) Write down all the roots of $X^n - \alpha$. Deduce that $L = K(\beta)$. **[3]**

(c) Prove that $\mathrm{Gal}(K : F)$ and $\mathrm{Gal}(L : K)$ are both abelian. [You may assume that they are both groups.] **[9]**

End of Paper