Queen Mary
University of London

M. Sci. Examination by course unit 2014

MTH743U Complex Systems

Duration: 3 hours

Date and time: 30 April 2014, 14:30h–17:30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

This is an OPEN BOOK exam

permitted:	any printed material, e.g. books any handwritten notes photocopies of any kind
prohibited:	using electronic devices, e.g. calculators or mobile phones sharing material with other students

Exam papers must not be removed from the examination room.

Examiner(s): V. Latora

Question 1 Consider the map $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} x + 2/3 & \text{if} \quad 0 \le x \le 1/3 \\ -3x + 2 & \text{if} \quad 1/3 \le x \le 2/3 \\ 2(x - 2/3) & \text{if} \quad 2/3 \le x \le 1 \end{cases} .$$

a) Sketch the graph of the map. Show that the partition $\{I_0, I_1, I_2\}$ with $I_0 = [0, 1/3]$, $I_1 = [1/3, 2/3]$ and $I_2 = [2/3, 1]$ is a Markov partition, and write down the corresponding topological transition matrix. [6 marks]

b) Show that the map is expanding. Show that the map is a piecewise linear Markov map. [5 marks]

c) Determine all admissible periodic symbol sequences of period $p = 2$. Compute the number N_p of periodic points of period $p = 1, 2, 3$ of the map, and the asymptotic behavior of N_p for large p. [11 marks]

d) Determine the transfer matrix of the map. Compute the invariant density.

[8 marks]

e) Assuming ergodicity, compute the Lyapunov exponent of the map. Explain why ergodicity is important. [5 marks]

Question 2 Consider the map $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = 1 - 2x^2$.

a) Write down the Frobenius-Perron equation of the map. [12 marks]

b) Which one of the two functions:

$$\rho(x) = C\frac{1}{\sqrt{1 - x^2}} \qquad \text{and} \qquad \rho(x) = C(1 + x^2)$$

is a solution of the Frobenius-Perron equation of the map. [6 marks]

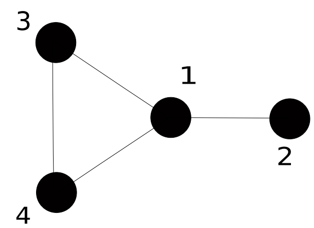
c) In the case in which $\rho(x)$ is a solution of the Frobenius-Perron equation, determine the normalisation constant C such that ρ is the density of a probability measure on $[-1, 1]$. [7 marks]

d) Assuming that the density $\rho(x)$ gives rise to an ergodic invariant measure, compute the probability that the points of a trajectory are in the interval $[-1, 0]$. [5 marks]

Question 3 Consider the following equations of motion:

$$\dot{x}_i(t) = f(x_i(t)) + \sigma \sum_j G_{ij} h(x_j(t))$$

describing the dynamics of a coupled network. Here, $x_i(t)$ denotes the state at node i, $f(x) = 4x(1-x)$ governs the local dynamics at a node, $h(x) = x$ determines the coupling, $\sigma \in \mathbb{R}$ denotes the coupling strength, and G_{ij} the Laplacian of the underlying graph. The network consists of $N = 4$ nodes organized in the following configuration:



a) Compute the degree distribution, the average distance between nodes, and the diameter of the graph. [8 marks]

b) Compute the master stability function for each of the time-independent synchronised states. [11 marks]

c) Find the eigenvalues of the Laplacian of the network. [8 marks]

d) For each synchronised state find the values of σ such that the state is transversely stable. [8 marks]

End of Paper