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Masters Degrees in Mathematics by Course Units

22 May 2014 Start: 10 am

MTH742P Advanced Combinatorics Duration: 3 hours

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Except for the matter of a mere pass, only the 4 best questions will be counted

CALCULATORS ARE NOT PERMITTED IN THIS EXAMINATION.

COMPLETE ALL ROUGH WORKINGS IN THE ANSWER BOOK AND CROSS THROUGH ANY WORK WHICH IS NOT TO BE ASSESSED.

IMPORTANT NOTE:
THE ACADEMIC REGULATIONS STATE THAT POSSESSION OF UNAUTHORISED MATERIAL AT ANY TIME WHEN A STUDENT IS UNDER EXAMINATION CONDITIONS IS AN ASSESSMENT OFFENCE AND CAN LEAD TO EXPULSION FROM QMUL.

PLEASE CHECK NOW TO ENSURE YOU DO NOT HAVE ANY NOTES, MOBILE PHONES OR UNATHORISED ELECTRONIC DEVICES ON YOUR PERSON. IF YOU HAVE ANY THEN PLEASE RAISE YOUR HAND AND GIVE THEM TO AN INVIGILATOR IMMEDIATELY. PLEASE BE AWARE THAT IF YOU ARE FOUND TO HAVE HIDDEN UNAUTHORISED MATERIAL ELSEWHERE, INCLUDING TOILETS AND CLOAKROOMS IT WILL BE TREATED AS BEING FOUND IN YOUR POSSESSION. UNAUTHORISED MATERIAL FOUND ON YOUR MOBILE PHONE OR OTHER ELECTRONIC DEVICE WILL BE CONSIDERED THE SAME AS BEING IN POSSESSION OF PAPER NOTES. MOBILE PHONES CAUSING A DISRUPTION IS ALSO AN ASSESSMENT OFFENCE.

EXAM PAPERS CANNOT BE REMOVED FROM THE EXAM ROOM.

Examiner: Prof. T. W. Müller

Question 1

a) State and prove the (discrete) *binomial theorem.*

[8 mark(s)]

b) Use the binomial theorem to establish the *Chu-Vandermonde Convolution Identity*

$$\sum_{i\geq 0} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k},$$

where k, m, n are integers with $0 \leq k \leq m+n$.

[7 mark(s)]

c) Given a finite set S with $n > 0$ elements, show that the number of subsets of S of odd size equals the number of subsets of even size.

[10 mark(s)]

Question 2

a) You may assume the combinatorial definition of binomial coefficients (i.e., $\binom{n}{k}$ is the number of k-element subsets of an n-set), as well as the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for $0 \le k \le n$.
For non-negative integers n, k, and $n_1, n_2, \dots, n_k$ with $n_1 + n_2 + \cdots + n_k = n$, denote by

$$\binom{n}{n_1, n_2, \dots, n_k}$$

the number of ways to distribute n elements into k consecutive boxes, such that, for each j with $1 \le j \le k$, the j-th box contains precisely n_j elements. (These numbers are called multinomial coefficients.) Show that

$$\binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \, \cdots \, n_k!}.$$

[6 mark(s)]

b) Let R be a commutative ring with identity element 1. For a positive integer k and arbitrary elements $x_1, \dots, x_k \in R$, prove *the multinomial theorem*

$$(x_1 + \cdots + x_k)^n = \sum_{\substack{n_1, \dots, n_k \ge 0 \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \dots, n_k} x_1^{n_1} \cdots x_k^{n_k}.$$

[7 mark(s)]

c) Show: for given integers n, k with $1 \le k \le n$, there is exactly one partition $n = n_1 + \cdots + n_k$ of n maximizing the multinomial coefficient $\binom{n}{n_1, \dots, n_k}$, namely the partition

$$n = (k-r)\left\lfloor \frac{n}{k} \right\rfloor + r\left(\left\lfloor \frac{n}{k} \right\rfloor + 1\right),$$

where $0 \le r < k$ and $r \equiv n \bmod k$. Conclude that

$$\max_{\substack{(n_1, \dots, n_k) \in \mathbb{N}_0^k \\ n_1 + \cdots + n_k = n}} \binom{n}{n_1, \dots, n_k} = \frac{n!}{\left(\lfloor \frac{n}{k} \rfloor !\right)^{k-r} \left(\lfloor \frac{n}{k} \rfloor + 1\right)!^r}$$

with r as above.

[12 mark(s)]

Question 3

a) Define *incidence structures*, *left-tactical* and *right-tactical configurations*, as well as *tactical configurations*. What are the *parameters* of a tactical configuration?

[7 mark(s)]

b) State and prove the standard identity (in terms of its parameters) associated with a tactical configuration. What identity can be proved if the incidence structure is left-tactical, but not right-tactical?

[8 mark(s)]

c)

(i) Explain what is meant by a 2-*design*. What are its *parameters*?

(ii) State the two identities associated with a 2-design, which are expressed in terms of its parameters.

(iii) Define the *localisation* $\mathfrak{I}_P$ of an incidence structure $\mathfrak{I} = (\mathfrak{P}, \mathfrak{B}, I)$ at a point $P \in \mathfrak{P}$. Explain briefly the connection between this construction and the standard identities for 2-designs.

[10 mark(s)]

Question 4

a)

(i) Let Ω be a set, and let Γ be a group. Explain what it means that Γ *acts* on the set Ω.

(ii) In the context of Part (i), what does it mean that Γ *acts freely* on Ω?

(iii) Suppose that G is a finite group acting freely on a finite set Ω. Show that $|G|$ divides $|\Omega|$.

[10 mark(s)]

b) Let Γ be a finitely generated group, let $s_n(\Gamma)$ be the number of subgroups of index n in Γ, and let $t_n(\Gamma)$ be the number of *transitive* permutation representations $\varphi : \Gamma \to S_n$ of Γ of degree n.

(i) State a quantitative relation between these two sequences of numbers.

(ii) Outline briefly the strategy of proof of the relation in Part (i); in particular, explain the connection with free group actions.

[10 mark(s)]

c) As in Part b), let Γ be a finitely generated group, let $h_n(\Gamma) = |\mathrm{Hom}(\Gamma, S_n)|$ be the number of permutation representations of Γ of degree n, and let $t_n(\Gamma)$ be as in Part b). Apply the exponential formula (with justification) to prove that

$$\sum_{n\geq 0} h_n(\Gamma) z^n/n! = \exp\left(\sum_{n\geq 1} t_n(\Gamma) z^n/n!\right). \tag{1}$$

Combining (1) with Part b) (i), deduce the formal identity

$$\sum_{n\geq 0} h_n(\Gamma) z^n/n! = \exp\left(\sum_{n\geq 1} s_n(\Gamma) z^n/n\right). \tag{2}$$

[5 mark(s)]

Question 5

Consider the formal identity

$$\sum_{n\geq 0}(n!)^{r-1}z^n = \exp\left(\sum_{n\geq 1} s_n(F_r)z^n/n\right), \tag{3}$$

where F_r is the free group of rank $r \geq 1$, and $s_n(F_r)$ denotes the number of subgroups of index n in F_r.

a) Use (3) to deduce the recurrence relation

$$s_n(F_r) = n(n!)^{r-1} - \sum_{k=1}^{n-1} s_k(F_r)((n-k)!)^{r-1}, \quad n \geq 1$$

for the sequence $(s_n(F_r))_{n\geq 1}$ (this relation is due to M. Hall Jr.).

[6 mark(s)]

b) Show that, for all integers n, r with $n, r \geq 1$, we have

$$s_n(F_r) \equiv 1 \mod 2.$$

[5 mark(s)]

c) Prove that, for $r \geq 2$,

$$s_n(F_r) \sim n(n!)^{r-1} \quad (n \to \infty).$$

[14 mark(s)]

End of Paper