Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH6142 / MTH6142P: Complex networks

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: L. Lacasa, G. Bianconi & V. Latora

Question 1 [15 marks].

Definitions and Examples of real-world networks.

(a) Define what a weighted undirected network is. Give an example of a real-world weighted undirected network, where you explain (i) what the nodes represent, (ii) what the links represent, (iii) what the weights represent. **[5]**

(b) Define what a bipartite network is. Give an example of a real-world bipartite network, where you explain (i) what each non-overlapping node set represents, (ii) what the links represent. **[5]**

(c) Consider the 'importance' of the nodes of a complex network. Give (i) an example of a real-world network and a process where the importance of a node is better measured in terms of degree centrality, and (ii) another example where the importance is better measured in terms of betweenness centrality. **[5]**

Question 2 [30 marks].

Structural properties of a given network.
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 4$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

(a) Draw the network. Is the network directed or undirected? (Explain your answer.) **[5]**

(b) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}\}$. **[5]**

(c) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

(d) Calculate the eigenvector centrality x_i of each node $i = 1, 2, \ldots, N$ of the network with adjacency matrix $\mathbf{A}$ defined above.
To this end, start from the initial guess $\mathbf{x}^{(0)} = \frac{1}{N}\mathbf{1}$ where $\mathbf{1}$ is the N-dimensional column vector of elements $1_i = 1$ for $i = 1, 2, \ldots, N$. Consider the iteration

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)},$$

for $n \in \mathbb{N}$.
Finally calculate the eigenvector centrality x_i of each node i of the network by finding the limit

$$x_i = \lim_{n\to\infty} \frac{x_i^{(n)}}{\sum_{j=1}^N x_j^{(n)}}$$

if $\lim_{n\to\infty} \sum_{j=1}^N x_j^{(n)} > 0$, otherwise if $\lim_{n\to\infty} \sum_{j=1}^N x_j^{(n)} = 0$ set $x_i = 0$ for $i = 1, 2, \ldots, N$. **[8]**

(e) Calculate the Katz centrality x_i of each node $i = 1, 2, \ldots, N$ of the network, where

$$x_i = \alpha \sum_{j=1}^{N} A_{ij} x_j + \beta,$$

with $\alpha \in (0, 1/\lambda_1)$ and λ_1 is the maximum eigenvalue of $\mathbf{A}$ and $\beta > 0$. **[8]**

Question 3 [25 marks].

Giant component of the random graph.
Consider a random graph ensemble $\mathbb{G}(N, p)$ formed by all networks of N nodes with each pair of nodes connected with probability p.
Take

$$p = \frac{c}{N-1}$$

with $c > 0$ indicating the average degree of the network.
Let S indicate the probability that a node is in the giant component.
A node i is not in the giant component of a random graph if for every other node j of the graph either one of the following events occurs:

(i) i is not connected to j by a link,

(ii) i is linked to j but j doesn't belong to the giant component.

(a) Show that in the large network limit $N \gg 1$, the probability S satisfies the equation

$$S = 1 - e^{-cS},$$

where c is assumed to be independent of the network size N. **[8]**

(b) Using graphical arguments, justify that in the large network limit $N \gg 1$, the network will have a non-null giant component if and only if $c > 1$. **[5]**

(c) Consider the case in which p is given by $p = 0.5$. Does the random graph $\mathbb{G}(N, p)$ have a giant component in the limit $N \to \infty$? (Why?) **[2]**

(d) Consider the case in which p is given by $p = 0.5/N^{2.5}$. Does the random graph $\mathbb{G}(N, p)$ have a giant component in the large network limit $N \to \infty$? (Why?) **[2]**

(e) Show that the average degree c that ensures that the random network is connected, i.e. such that there are no isolated nodes, is approximately given by

$$c \simeq \ln(N).$$ **[8]**

Question 4 [15 marks].

Power-law networks
Consider a power-law network with N nodes. Suppose the degree distribution is $p(k) = ck^{-\gamma}$ with exponent $\gamma > 1$, and the smallest and largest degree are respectively equal to k_{min} and k_{max}. In the following, work in the so-called continuous-k approximation, i.e. treat the degree k as a real positive number.

(a) Determine the value of the normalisation constant c. **[5]**

(b) Find an expression for the average node degree, $\langle k \rangle$, and an expression for the second order moment of the degree distribution, $\langle k^2 \rangle$. **[5]**

(c) What is the main difference between a Poisson and a scale-free network? Why are scale-free networks in general better models of real-world networks than Poisson networks? **[5]**

Question 5 [15 marks].

Complex networks
Complex networks are neither completely ordered (like lattices) nor completely disordered (like random graphs). Using your own words:

(a) Discuss the main properties of real-world complex networks such as social networks or the Internet, **[5]**

(b) Discuss how these resemble or differ from classic synthetic models such as lattices or random graphs, and why the differences are so important in, for instance, epidemic spreading. **[5]**

(c) Discuss also what other synthetic models beyond lattices or random graph models have been proposed recently to be able to recover the properties we observe in real-world networks. **[5]**

End of Paper.