Queen Mary
University of London

Main Examination period 2017

MTH6142 / MTH6142P
Complex networks

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: G. Bianconi & V. Latora

Question 1. [30 marks]

Structural properties of a given network.
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 4$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

a) Is the network directed or undirected? (*Justify your answer*) **[2]**

b) Draw the network. **[4]**

c) How many weakly connected components are there in the network? Which are the nodes in each weakly connected component? **[2]**

d) How many strongly connected components are there in the network? Which are the nodes in each strongly connected component? **[2]**

e) Is there an out-component in the network? If yes, indicate the nodes in the out-component of the network. **[2]**

f) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}\}$. **[4]**

g) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

h) Calculate the eigenvector centrality x_i of each node $i = 1, 2, \ldots, N$ of the network with adjacency matrix $\mathbf{A}$ given by Eq. (1).
To this end start from the initial guess $\mathbf{x}^{(0)} = \frac{1}{N}\mathbf{1}$ where $\mathbf{1}$ is the N-dimensional column vector of elements $1_i = 1\ \forall i = 1, 2 \ldots, N$.
Consider the iteration

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)}$$

for $n \in \mathbb{N}$.
Finally evaluate the eigenvector centrality x_i of each node i of the network by calculating the limit

$$x_i = \lim_{n\to\infty} \frac{x_i^{(n)}}{\sum_{j=1}^{N} x_j^{(n)}}.$$

[8]

i) Is the result obtained in point h) expected? (*Why?*) **[2]**

Question 2. [35 marks]

Diameter and clustering coefficient of networks

a) Which is the undirected network of N nodes with smallest diameter?
Does this network have the small-world distance property? *Why?* **[5]**

b) Which is the undirected network of N nodes which is connected and has the largest diameter?
Does this network have the small-world distance property? *Why?* **[5]**

c) Consider a random Poisson network with average degree $\langle k\rangle = 4$ and total number of nodes N.
Indicate with ℓ the average shortest path in the network.

i) Using the properties of the generating function evaluate the average branching ratio of a node reached by following a link given by $\langle b\rangle = \frac{\langle k(k-1)\rangle}{\langle k\rangle}$. **[6]**

ii) Approximate the number of nodes N_d at distance $d \geq 1$ from a random node of the network as $N_d = \langle k\rangle \left(\frac{\langle k(k-1)\rangle}{\langle k\rangle}\right)^{d-1}$ and show that $N_d = 4^d$. **[3]**

iii) Using the properties of the geometric sum, evaluate the total number $N_{d\leq\ell}$ of nodes at distance $0 \leq d \leq \ell$ from a random node of the network. **[4]**

iv) Impose that all the nodes of the Poisson network can be found within a distance $d \leq \ell$ from any random node. Using the result obtained in (ii), express the average distance ℓ of the Poisson network in terms of the total number of nodes N. **[4]**

v) Consider the expression found in (iii).
Find the leading term of ℓ in terms of the total number of nodes N in the network, when $N \gg 1$. **[4]**

vi) Estimate the local clustering coefficient of a generic node of the network. **[4]**

Question 3. [35 marks]

Growing network model

Consider the following model for a growing simple network.
We adopt the following notation: N and L indicate respectively the total number of nodes and links of the network, A_{ir} indicates the generic element of the adjacency matrix $\mathbf{A}$ of the network and k_i indicates the degree of node i.
At time $t = 0$ the network is formed by a $n_0 = 2$ nodes and a single link (initial number of links $m_0 = 1$) connecting the two nodes.
At every time step $t > 0$ the network evolve according to the following rules:

- A single new node joins the network.
- A link (i, r) between a node i and a node r is chosen randomly with uniform probability

$$\pi_{(i,r)} = \frac{A_{i,r}}{L}$$

  and the new node is linked to both node i and node r.

a) Show that in this network evolution at each time step the average number of links $\tilde{\Pi}_i$ added to node i follows the preferential attachment rule, i.e.

$$\tilde{\Pi}_i = \sum_{r=1}^{N} \pi_{(i,r)} = 2\frac{k_i}{\sum_{j=1}^{N} k_j}.$$

[6]

b) What is the total number of links in the network at time t? What is the total number of nodes? **[2]**

c) What is the average degree $\langle k \rangle$ of the network at time t? **[4]**

d) Use the result at point a) to derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i for $t \gg 1$ in the mean-field, continuous approximation. **[6]**

e) What is the degree distribution of the network at large times in the mean-field approximation? **[6]**

f) Let $N_k(t)$ be the average number of nodes with degree k at time t. Write the master equation satisfied by $N_k(t)$. **[3]**

g) Solve the master equation, finding the exact result for the degree distribution $P(k)$ in the limit $N \to \infty$. **[8]**

End of Paper.