Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH6141 Random Processes

Duration: 2 hours

Date and time: 3rd June 2014, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

Each question carries 20 marks. You should attempt ALL FIVE questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): Mark Jerrum

Question 1 Parts (a)–(c) of this question concern the Markov chain $X_0, X_1, X_2, \dots$ with state space $S = \{1, 2, 3, 4, 5\}$, and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Define the term *absorbing state*. Which are the absorbing states in the given Markov chain? [3]

(b) Let a_i be the probability of absorption in state 5, given that the Markov chain starts in state i. By conditioning on X_1 ("first-step analysis") write down equations satisfied by a_i. Explain how you arrived at the equation involving a_1. [8]

(c) Hence compute the probability of absorption in state 5 given that the Markov chain starts in state 1. [3]

(d) In general, the equations derived from first-step analysis may not have a unique solution. This happens when there is some state from which it is impossible to reach any absorbing state. Suggest a way to modify the first-step analysis (as used above) to calculate absorption probabilities in this situation. Illustrate your answer with reference to the Markov chain on state space S as before, but with transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Note that it is impossible to reach the sole absorbing state 5 from either state 2 or state 4.) What is the probability of eventual absorption in state 5, given that the Markov chain starts in state 1? [6]

Question 2 Let P be the transition matrix of a Markov chain on a finite state space S.

(a) Explain what it means for P to be (i) *irreducible* and (ii) *regular*. [4]

(b) What does it mean for probability vector $\boldsymbol{w}$ to be an *equilibrium distribution* for P? [2]

(c) Suppose P is irreducible and define $Q = \frac{1}{2}(I + P)$, where I is the identity matrix. Prove that Q is regular, and has the same (unique) equilibrium distribution as P. [5]

Now specialise P to be the transition matrix of the Ehrenfest Urn Model with three balls:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(d) What is the equilibrium distribution of P? Does P have a limiting distribution? Justify your answer. [6]

(e) Let Q be the transition matrix of the following "lazy" version of the Ehrenfest Urn Model (with three balls). At each transition of the lazy model we flip a fair coin. If it comes up heads, we do nothing; if it comes up tails we move a ball according to the usual rule. What is the equilibrium distribution of Q? Does Q have a limiting distribution? Justify your answer. [3]

Question 3 This question concerns a discrete-time Markov chain $X_0, X_1, X_2, \ldots$ on state space $\mathbb{N}$.

(a) Define the *return probability* f_{ii} for a state i, and say what it means for state i to be *recurrent*. [3]

(b) Give a condition, in terms of the t-step transition probabilities $p_{ii}^{(t)}$, for state i to be recurrent. [3]

Now let $0 < p < 1$ and suppose that the transition probabilities of the Markov chain are given by $p_{i,i+1} = p$, $p_{i,0} = 1 - p$ for all $i \in \mathbb{N}$ (with all other probabilities being 0).

(c) Calculate $\Pr(X_2 = 2 \mid X_0 = 0)$ and $\Pr(X_2 = 0 \mid X_0 = 0)$. [4]

(d) Calculate $p_{0,0}^{(t)}$ for all $t \geq 1$, and decide whether state 0 is recurrent. [4]

(e) Does the Markov chain have an equilibrium distribution? [2]

(f) Calculate $p_{1,1}^{(t)}$ for all $t \geq 1$. [4]

Question 4 Let $(X(t) : t \geq 0)$ be a Poisson process of rate λ.

(a) State the distribution of $X(t) - X(s)$ for given s, t with $0 \leq s < t$. [3]

(b) Calculate the following probabilities, showing your working or reasoning. [8]

(i) $\Pr(X(3) = 4)$,

(ii) $\Pr(X(3) = 4 \mid X(2) = 3)$,

(iii) $\Pr(X(2) = 3, X(3) = 4)$,

(iv) $\Pr(X(3) = 3 \mid X(2) = 4)$.

(c) Suppose $0 < u \leq t$. Compute the conditional probability

$$\Pr(X(u) = 1 \mid X(t) = 1).$$

Hence deduce the distribution of the first arrival, conditioned on there being exactly one arrival in the interval $(0, t]$. [5]

(d) Customers arrive in a shop according to a Poisson process of rate λ per minute. Each customer leaves the shop after exactly 10 minutes. What is the conditional expectation of the number of customers in the shop after it has been open for one hour, given that n customers arrived in total during the hour. [4]

Hint. Use a generalisation of the fact you derived in part (c).

Question 5 (a) Explain what it means for a stochastic process $X(t)$ on $\mathbb{N}$ to be a *birth process.* [4]

The remainder of the question concerns a birth process $X(t)$ with (birth) parameters $\lambda_n = n$ and initial state $X(0) = 1$.

(b) Describe a plausible situation in which such a birth process (or an approximation to it) might arise. [2]

(c) For $n \geq 1$ define $p_n(t) = \Pr(X(t) = n)$. Show that the functions $p_n(t)$ satisfy the equations

$$p_n'(t) = (n-1)p_{n-1}(t) - np_n(t),$$

for all $n \geq 1$. [5]

(d) (i) Use the equations from part (c) to find $p_1(t)$.

(ii) Use the equations to find $p_2(t)$.

(iii) Hence determine the probability that there are at least 3 individuals at time 1. [9]

End of Paper