MTH6140 / MTH6140P: Linear Algebra II

Duration: 2 hours

Date and time: 25th May 2016, 14:30–16:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): M. Jerrum

Question 1. In this question, V is a finite-dimensional vector space over a field $\mathbb{K}$.

(a) Suppose $u, v \in V$ are vectors and $a, b \in \mathbb{K}$ are scalars. Two of the following six expressions are invalid. Which are they?

$$ab, \quad au, \quad uv, \quad a+b, \quad a+u, \quad u+v.$$

No explanation is required. **[2]**

(b) Explain what it means for a list of vectors in V to be (i) *linearly independent*, (ii) *spanning*, and (iii) a *basis*. **[5]**

(c) Suppose that $(v_1, \ldots, v_m)$ is a list of vectors that spans V. Show that if the list is linearly dependent then it is possible to remove one vector from the list so that what remains also spans V. **[5]**

(d) Deduce that every list of vectors that spans V contains a basis of V. **[3]**

(e) The following list of vectors spans $\mathbb{R}^3$:

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad w_3 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \quad w_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad w_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Illustrate your answer to parts (c) and (d) by identifying a basis of $\mathbb{R}^3$ from within this list. **[5]**

Question 2. This question concerns $n \times n$ matrices over a field $\mathbb{K}$.

(a) In this part only, set $n = 3$. Write down the elementary matrices corresponding to the elementary row operations of (i) adding row 3 to row 1, (ii) interchanging rows 1 and 2, and (iii) multiplying row 2 by the scalar $c \in \mathbb{K}$. **[3]**

(b) Let A be an $n \times n$ matrix. Describe how $\det(A)$ changes when (i) one row of A is added to another, (ii) two rows of A are interchanged, and (iii) one row of A is multiplied by a scalar $c \in \mathbb{K}$. (No justification is required.) **[3]**

(c) Recall that a non-singular matrix may be reduced to the identity matrix by applying a sequence of elementary row operations (i.e., multiplying on the left by elementary matrices). Let A and B be non-singular matrices. Prove that $\det(AB) = \det(A)\det(B)$. Does this identity also hold when either A or B is singular? **[7]**

(d) Define the relation of *similarity* between matrices. **[3]**

(e) Prove that similar matrices have the same determinant. **[4]**

Question 3. Suppose V and W are vector spaces over a field $\mathbb{K}$.

(a) Explain what it means for α to be a *linear map* from V to W. **[3]**

(b) Define the *kernel* $\mathrm{Ker}(\alpha)$ and *image* $\mathrm{Im}(\alpha)$ of the linear map α. **[4]**

(c) Take a basis $(v_1, \ldots, v_k)$ for $\mathrm{Ker}(\alpha)$, and extend it to a basis $(v_1, \ldots, v_n)$ for V. Prove that the vectors $\alpha(v_{k+1}), \ldots, \alpha(v_n)$ span $\mathrm{Im}(\alpha)$. **[6]**

(d) State, without proof, a relation between $\dim(V)$, $\dim(\mathrm{Ker}(\alpha))$ and $\dim(\mathrm{Im}(\alpha))$. **[3]**

(e) Suppose $\alpha : V \to W$ and $\beta : W \to V$ are linear maps. Prove that $\dim(\mathrm{Ker}(\beta\alpha)) \geq \dim(V) - \dim(W)$. **[4]**

Question 4.

(a) Define the *characteristic polynomial* and the *minimal polynomial* of a linear map α on a vector space. Briefly explain why these definitions make sense. **[6]**

(b) A linear map α on $\mathbb{R}^3$ is represented with respect to some basis by the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 5 & -2 \\ 3 & 6 & -2 \end{bmatrix}.$$

Compute the characteristic and minimal polynomials of A. **[9]**

(c) Is A diagonalisable? Explain your answer. **[2]**

(d) If the answer to part (c) is "yes", then write down a diagonal matrix that is similar to A. If the answer is "no", write down a matrix in Jordan form that is similar to A. **[3]**

Question 5. In this question, V is an inner product space over $\mathbb{R}$, and α is a linear map on V.

(a) Let U be a subspace of V. Define the *orthogonal complement*, $U^\perp$, of U. **[3]**

(b) Prove that $U^\perp$ is a subspace of V. **[3]**

(c) State, without proof, the relationship between $\dim(U)$, $\dim(U^\perp)$ and $\dim(V)$. **[2]**

(d) Find a basis for the orthogonal complement $U^\perp$ of the subspace U of $\mathbb{R}^4$ spanned by **[5]**

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Show your working.

(e) Explain what it means for α to be *self-adjoint*. **[3]**

(f) Suppose v and w are eigenvectors of α with distinct eigenvalues λ and μ. Assuming α is self-adjoint, show that $v \cdot w = 0$. **[4]**

End of Paper.