Queen Mary University of London

Main Examination period 2023 – January – Semester A

MTH6132 / MTH6132P: Relativity

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: Pau Figueras, Juan Valiente Kroon

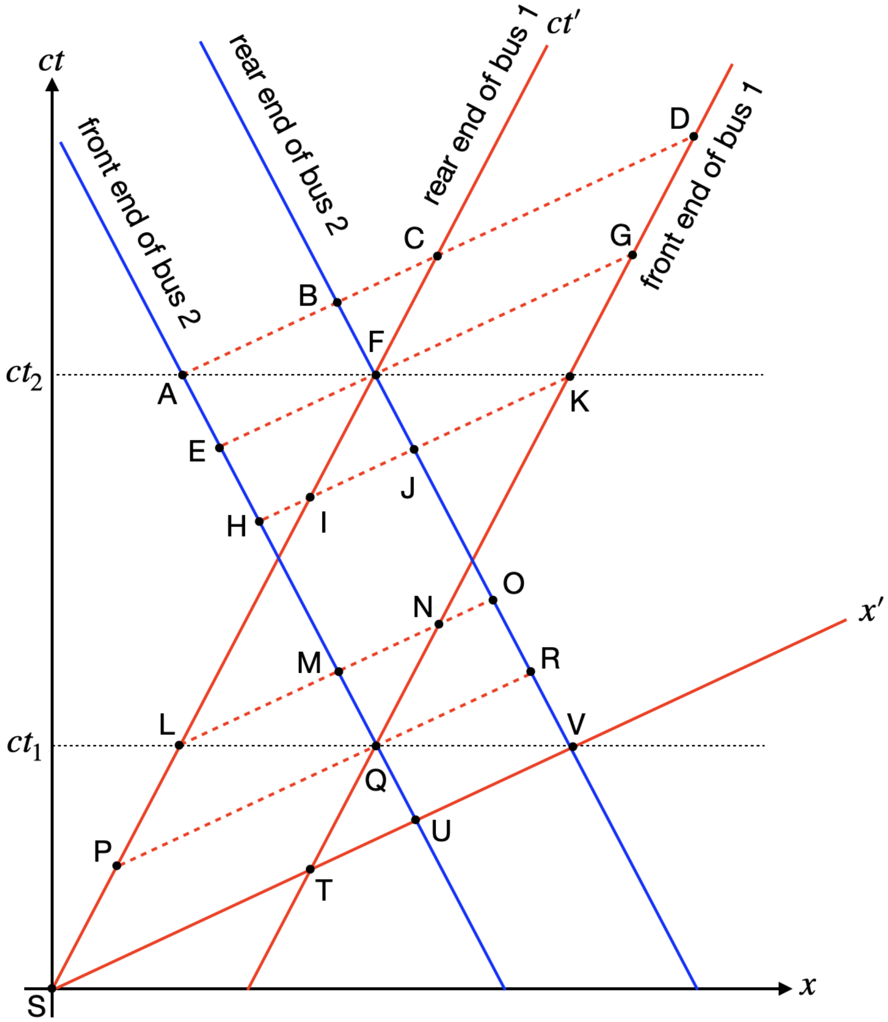

Figure 1: Spacetime diagram for Question 1. The axes (x, ct) correspond to the reference frame of the tunnel, while the axes (x', ct') correspond to the reference frame of bus 1.

Question 1 [20 marks].

Two identical buses, labelled 1 and 2, are traveling at the same speed in opposite directions inside a long and straight tunnel. The tunnel workers are on strike and, as a consequence, the tunnel lights are off. At time t_1 in the reference frame of the ground, all the ceiling lights in the tunnel turn on simultaneously. Then, at a later time t_2 in the reference frame of the ground, the tunnel workers resume their strike action and turn off all the lights simultaneously.

A scientist on the ground concludes that the two buses in the tunnel are hit by the same number of photons since they have the same length and they immersed in light for the same amount of time. However, another scientist travelling on bus 1 reasons that since bus 1 is longer than bus 2 in the reference frame of bus 1, bus 1 must be hit by more photons than bus 2. The two scientists seem to contradict each other. Answer the following questions referring to the spacetime diagram in Fig. 1 before a fight between the two scientists breaks out.

(a) At which spacetime points, labelled A through V, are the front and rear ends of bus 1 when its front end enters the field of light in bus' 1 reference frame? [**4**]

(b) At which spacetime points, labelled A through V, are the front and rear ends of bus 1 when its rear end exits the field of light in bus' 1 reference frame? [**4**]

(c) At which spacetime points, labelled A through V, are the front and rear ends of bus 2 when its rear end enters the field of light in bus'1 reference frame? [**4**]

(d) At which spacetime points, labelled A through V, are the front and rear ends of bus 2 when its front end exits the field of light in bus'1 reference frame? [**4**]

(e) What is wrong with the reasoning of the scientist on bus 1? [**4**]

Question 2 [20 marks]. Consider two vector fields X and Y and an arbitrary smooth scalar function $f(x)$. The Lie derivative of f along X is given by $\mathcal{L}_X(f) = X(f) = X^a\partial_a f$. Similarly, in the lectures we saw that the Lie derivative of Y along X, written as $\mathcal{L}_X Y$, is given by the commutator $[X, Y]$. From the definition of $[X, Y]$ acting on f as in the notes,

$$[X, Y](f) \equiv X(Y(f)) - Y(X(f))\,,$$

(a) Show that the components of the vector field $[X, Y]^a$ are [**5**]

$$[X, Y]^a = X^b\partial_b Y^a - Y^b\partial_b X^a\,.$$

(b) Show that $[X, Y]^a$ transforms as a vector under general coordinate transformations. [**5**]

(c) Consider a one form ω_a. Using that the Lie derivative satisfies the Leibniz rule and recalling that $\omega_a Y^a$ is a scalar, show that

$$\mathcal{L}_X(\omega)_a = X^b\partial_b\omega_a + \omega_b\partial_a X^b\,.$$

[**5**]

(d) Is $\mathcal{L}_X(\omega)_a$ a one form? [**5**]

Question 3 [20 marks]. Consider the following line element:

$$ds^2 = dr^2 + \tanh^2(r)\, d\phi^2\,, \tag{1}$$

with $r > 0$ and $\phi \sim \phi + 2\pi$.

(a) Compute the Christoffel symbols. **[5]**

(b) Show that the non-vanishing components of the Ricci tensor are given by

$$R_{rr} = \frac{2}{\cosh^2(r)}\,, \quad R_{\phi\phi} = \frac{2\,\tanh^2(r)}{\cosh^2(r)}\,,$$ **[10]**

(c) Does the line element (1) solve the Einstein equation in vacuum $G_{ab} = 0$? **[5]**

Question 4 [20 marks]. Consider the following spacetime:

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} - M} + r^2 d\phi^2\,, \tag{2}$$

where $\ell > 0$ and $M > 0$ are constants.

(a) Find the effective potential governing the trajectories of timelike and null geodesics. **[5]**

(b) Sketch the effective potential and describe the possible trajectories of the radial null geodesics. **[5]**

(c) Does this spacetime contain a black hole? Reason your answer. **[5]**

(d) Find coordinates such that the metric (2) is regular at $r = \ell\, M$. **[5]**

Question 5 [20 marks].

(a) Explain in your own words and in a concise way (no more than a short paragraph) what gravitational waves are. **[5]**

(b) Show that the non-vanishing components of the quadrupole moment tensor I^{ij} for four particles of masses m_1 located at the points $(a, 0, 0)$ and $(-a, 0, 0)$, and masses m_2 located at $(0, a, 0)$ and $(0, -a, 0)$ respectively are given by: **[5]**

$$I^{xx} = 2\, m_1\, a^2\,, \quad I^{yy} = 2\, m_2\, a^2\,.$$

(c) Compute the components of the quadrupole moment tensor in frame in which the particles are rotating about the z axis on circle of radius a and angular velocity ω by considering the following coordinate transformation: [**5**]

$$\begin{aligned} x' &= x\,\cos(\omega\,t) - y\,\sin(\omega\,t) \\ y' &= x\,\sin(\omega\,t) + y\,\cos(\omega\,t) \\ z' &= z\,. \end{aligned}$$

(d) Use the quadrupole formula

$$P = \frac{G}{5}\langle \dddot{Q}_{i'j'}\,\dddot{Q}^{i'j'}\rangle$$

to compute the power radiated in gravitational waves by the rotating masses. (*Hint:* recall that $\langle\cos^2(\omega\,t)\rangle = \langle\sin^2(\omega\,t)\rangle = \frac{1}{2}$) [**5**]

End of Paper.