Main Examination period 2020 – January – Semester A

MTH6132: Relativity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: S. Majid, J. Valiente Kroon

You may refer to formulae in the appendix as well as general results from lectures.

Question 1 [16 marks]. In this question c denotes the speed of light.

(a) A muon at the origin of an inertial frame F moves along the x-axis at speed $v = 0.5c$ and decays after time $t = 10^{-2}$s. How much time elapsed in the muon's rest frame? **[5]**

(b) A rocket of length 100 metres in its rest frame travels at speed $v = 0.5c$ along the x-axis of an inertial frame F. What is the length of the rocket in frame F ? **[5]**

(c) Sketch a spacetime diagram for part (b) indicating the world-lines of the front and back of the rocket in frame F, the 100 metre rocket rest length and the rocket length in F. **[6]**

Question 2 [17 marks]. Let $\overline{A} = (A^0, \underline{A})$ and $\overline{B} = (B^0, \underline{B})$ be 4-vectors in Minkowski space, where $\underline{A}, \underline{B}$ are their respective 3-vector components.

(a) Define what it means for $\overline{A}$ to be **timelike** and for $\overline{B}$ to be **spacelike**. **[4]**

(b) Show that if $\overline{A}$ is timelike and $\overline{A} \cdot \overline{B} = 0$ with respect to the Lorentzian dot product then $\overline{B}$ is spacelike. **[8]**

(c) Define what it means for a 4-vector to be **null** and show that the sum of two null 4-vectors does not have to be null. **[5]**

Question 3 [16 marks]. Let $\overline{U}$ be the 4-velocity of a particle of rest mass m_0.

(a) Define the 4-momentum $\overline{p}$ of the particle in terms of $m_0, \overline{U}$. **[3]**

(b) Writing $\overline{p} = (Ec^{-1}, \underline{p})$, show that $E = m_0c^2$ for a particle at rest. **[4]**

(c) A particle of rest mass m_0 moving at $0.5c$ collides with and coalesces with a particle of mass $2m_0$ at rest in the laboratory frame. Find the rest mass and velocity of the resulting combined particle. You may assume that 4-momentum is conserved in the process. **[9]**

Question 4 [15 marks]. The components V^a of a vector on a manifold transform under a change of coordinates from x^a to x'^a as $V'^a = \frac{\partial x'^a}{\partial x^b} V^b$.

(a) Write down how the components W_a of a **covector** similarly transform. **[3]**

(b) Using the transformation properties of vectors and covectors, show that W_aV^a transforms as a **scalar**. **[6]**

(c) Using formulae for a covariant derivative ∇ in terms of Christoffel symbols $\Gamma^a{}_{bc}$, show that

$$(\nabla_a W_b)V^b + W_b(\nabla_a V^b) = \frac{\partial}{\partial x^a}(W_b V^b).$$ **[6]**

Question 5 [17 marks]. Let $A > 0$ be a constant. The metric for an expanding two-dimensional spacetime in certain units is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{At}\mathrm{d}x^2.$$

(a) Write down the covariant and contravariant components of the metric tensor for this spacetime. Use coordinates x^a where $x^0 = t, x^1 = x$. **[3]**

(b) Calculate the Christoffel symbols $\Gamma^a{}_{bc}$ for the Levi-Civita connection for this metric using the formula in the Appendix. **[9]**

(c) Use your results from part (b) to compute the curvature component $R^1{}_{010}$. **[5]**

Question 6 [19 marks]. This question concerns a Schwarzschild black hole in standard coordinates t, r, θ, φ and units where $c = 1$, with

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2).$$

(a) The metric is singular at two finite values of r. What are they and what is the key difference in the nature of the two singularities? **[4]**

(b) Show that a radially infalling geodesic with $\dot{\theta} = \dot{\varphi} = 0$ in the region $r > 2GM$ obeys

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = l \text{ (a constant)}, \quad l^2 = \dot{r}^2 + \left(1 - \frac{2GM}{r}\right). \qquad \textbf{[9]}$$

You may wish to use the relevant Christoffel symbols

$$\Gamma^0{}_{01} = \Gamma^0{}_{10} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}, \quad \Gamma^0{}_{00} = \Gamma^0{}_{11} = 0$$

and the normalisation of the 4-velocity.

(c) Show that the **proper time** for a particle to reach the event horizon coming in from $r = 8GM$ on the trajectory in part (b) with $l = 1$ is $\frac{28}{3}GM$. **[4]**

(d) What is the elapsed coordinate time t for the motion in part (c)? You are not asked to justify your answer. **[2]**

End of Paper – An appendix of 1 page follows.

You are reminded of the following, which you may use freely.

- Lower case Latin indices run from 0 to 3 for a four-dimensional spacetime.
- The metric tensor of the Minkowski spacetime is η_{ab} such that

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

- The Lorentz transformations between two frames F and F' in standard configuration with F' moving with speed v relative to F is

$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{v}{c^2}x\right), \quad y' = y, \quad z' = z, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

- 4-velocity in terms of usual velocity $\underline{v}$ with $v = |\underline{v}|$ and its normalisation

$$\overline{U} = \frac{\mathrm{d}\overline{x}}{\mathrm{d}\tau} = \gamma(v)(c, \underline{v}), \quad \overline{U}\cdot\overline{U} = -c^2.$$

- The covariant derivative of a covariant vector and contravariant respectively are given by

$$\nabla_a V_b = \partial_a V_b - V_c \Gamma^c{}_{ba}, \quad \nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ca} V^c.$$

- The metric tensor satisfies

$$g_{ab}g^{bc} = \delta_a{}^c.$$

- Christoffel symbols for the Levi-Civita connection:

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).$$

- The Riemann curvature tensor is given by

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}.$$

- Euler–Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial L}{\partial \dot{x^c}}\right) - \frac{\partial L}{\partial x^c} = 0.$$

- The geodesic equations are

$$\ddot{x^a} + \Gamma^a{}_{bc}\dot{x^b}\dot{x^c} = 0.$$

- The normalisation of 4-velocity in units where $c = 1$ is

$$g_{ab}\dot{x^a}\dot{x^b} = -1.$$

End of Appendix.