Queen Mary University of London

Main Examination period 2018

MTH6132: Relativity

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: H. T. Nguyen, J. A. Valiente-Kroon

Question 1. [15 marks] Let $\overline{A}, \overline{B}$ denote two arbitrary 4-vectors in Minkowski space-time.

(a) Define the scalar product $\overline{A} \cdot \overline{B}$. **[3]**

(b) State the definition for the **invariance** of $\overline{A} \cdot \overline{B}$. **[2]**

(c) Show that if $|\overline{A}|^2, |\overline{B}|^2$ and $|\overline{A} + \overline{B}|^2$ are invariant, then $\overline{A} \cdot \overline{B}$ is invariant. **[3]**

(d) Define **spacelike**, **timelike** and **null** vectors in Minkowski space. **[4]**

(e) Show that the sum of any two orthogonal spacelike vectors is also spacelike. **[3]**

Question 2. [16 marks]

(a) Let ϕ be a scalar and let $V_a = \frac{\partial \phi}{\partial x^a} = \partial_a \phi$. Show that V_a is tensorial. **[6]**

(b) Consider $\mathbb{R}^3$ with spherical co-ordinates (r, θ, φ) and line element given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2.$$

(i) Given the contravariant vector $X^a = (1, r, r^2)$, find X_a. **[5]**

(ii) Given the covariant vector $Y_a = (0, -r^2, r^2 \cos^2 \theta)$, find Y^a. **[5]**

Question 3. [16 marks] Let F and F' denote two inertial reference frames moving with velocity v with respect to each other.

(a) Given the Lorentz transformations in the appendix, compute the inverse Lorentz transformations. **[4]**

(b) Let $\Delta = -c^2 t^2 + x^2 + y^2 + z^2$. Show that Δ is invariant under Lorentz transformations. **[4]**

(c) Consider a particle moving along the x-axis. Its velocity in the x direction with respect to the frames F and F' are given, respectively, by

$$V = \frac{dx}{dt} \quad \text{and} \quad V' = \frac{dx'}{dt'}.$$

(i) Using the Lorentz transformation between F and F' show that

$$V' = \frac{V - v}{1 - Vv/c^2}.$$

[6]

(ii) State a formula for V in terms of V'. **[2]**

Question 4. [18 marks] Consider a space-time with metric

$$ds^2 = -e^{2Ar}dt^2 + dr^2.$$

(a) Write down the tensors g_{ab} and g^{ab} corresponding to this metric. **[4]**

(b) Compute the Lagrangian, L, of this metric. **[4]**

(c) Compute the Christoffel symbols and geodesic equations for this metric. **[10]**

Question 5. [15 marks]

(a) State the definition of the momentum 4-vector and the law of conservation of momentum. **[5]**

(b) In this question, assume that we are using units for which $c = 1$. A particle has rest mass m_0. While at rest, it emits a photon and its rest mass is reduced to $m_0/2$.

(i) Show that the speed of the particle after the reduction of mass is $3/5$. **[8]**

(ii) Compute the value of the energy of the photon, $E = h\nu$, in terms of m_0. **[2]**

Question 6. [20 marks] Consider the Schwarzschild metric, given in local co-ordinates (t, r, θ, φ) by

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

(a) If $A = 1 - \frac{2GM}{r}$ and $A' = \frac{2GM}{r^2}$ derive the geodesic equations for the metric in the form above (1) in terms of A, A'. **[8]**

(b) Describe which metric this is when $M = 0$. **[2]**

(c) Find the values of r for which this metric is singular. **[4]**

(d) Consider the following new co-ordinate system

$$\hat{t} = t + 2GM \ln|r - 2GM|, r = r, \theta = \theta, \varphi = \varphi.$$

Compute the line element in these co-ordinates. **[6]**

End of Paper – An appendix of 1 page follows.

- Lower case Latin indices run from 0 to 3
- The metric tensor of Minkowski space-time is η_{ab} where
$$ds^2 = \eta_{ab}dx^a dx^b = -c^2dt^2 + dx^2 + dy^2 + dz^2$$
- The Lorentz transformations between two frames F and F' in standard configuration are given by
$$x' = \gamma(x - vt), \quad t' = \gamma\left(t - \frac{vx}{c^2}\right), \quad y' = y, \quad z' = z, \quad \text{with } \gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$$
where F' is moving with speed v relative to F.
- A covariant vector is tensorial if
$$V'_a = \frac{\partial x^b}{\partial x'^a} V_b$$
and a contravariant vector is tensorial if
$$V'^a = \frac{\partial x'^a}{\partial x^b} V^b$$
- The covariant derivative of a covariant vector is given by
$$\nabla_a V_b = \partial_a V_b - \Gamma^c{}_{ab} V_c.$$
- The covariant derivative of a contravariant vector is given by
$$\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c.$$
- The metric tensor satisfies
$$g_{ab}g^{bc} = \delta^c_a.$$
- The Christoffel symbols (connection):
$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{da} - \partial_a g_{bd})$$
- The Riemann curvature tensor :
$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}$$
- Euler-Lagrange Equations:
$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^c}\right) - \frac{\partial L}{\partial x^c} = 0.$$
- Geodesic equations:
$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0.$$

End of Appendix.