Main Examination period 2020 – May/June – Semester B
Online Alternative Assessments

MTH6128 / MTH6128P: Number Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment, you may use books, notes, and the internet. You may use calculators and computers, but you should show your working for any calculations you do. You must not seek or obtain help from anyone else.

At the start of your work, please **copy out and sign** the following declaration:

> I declare that my submission is entirely my own, and I have not sought or obtained help from anyone else.

All work should be **handwritten**, and should **include your student number**.

You have **24 hours** in which to complete and submit this assessment. When you have finished your work:

- scan your work, convert it to a **single PDF file** and upload this using the upload tool on the QMplus page for the module;
- e-mail a copy to maths@qmul.ac.uk with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are not expected to spend a long time working on this assessment. We expect you to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the assessment period, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: S. Lester, S. Sasaki

Question 1 [26 marks].

(a) Define the terms **algebraic integer** and **quadratic integer**. State the Fundamental Theorem of Arithmetic. **[6]**

(b) Find the multiplicative inverse of $[17]_{71}$. Use your answer to find all integer solutions to the equation

$$17x \equiv 4 \pmod{71}.$$

Explain your working. **[5]**

(c) Determine which of the following numbers are quadratic integers. Explicitly state any results from the lectures that you use. **[6]**

(i) $\dfrac{2+\sqrt{52}}{4}$;

(ii) $\dfrac{\sqrt{43}}{2} - \dfrac{7}{2}$;

(iii) $\dfrac{\sqrt{25}}{2} - \dfrac{3}{2}$.

(d) Show that $\sqrt{3+\sqrt{11}}$ is an algebraic integer. **[5]**

(e) Determine all of the units in the ring $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Justify your answer. State any results from the lectures that you use. **[4]**

Question 2 [22 marks].

(a) Give an example of an irrational number whose continued fraction is not periodic. Explain why the number you gave has the desired properties. **[3]**

(b) Use the Euclidean algorithm to find a continued fraction expansion of $\dfrac{1723}{505}$. **[5]**

(c) Find the continued fraction expansion of $\dfrac{1+\sqrt{37}}{2}$. Explain your working. **[6]**

(d) Let $a_0, a_1, \ldots, a_n$ be positive integers. Let $c_k = p_k/q_k$ be the kth convergent of the continued fraction $[a_0; a_1, \ldots, a_n]$.

(i) Prove that, for each $1 \leqslant k \leqslant n$,

$$\frac{p_k}{p_{k-1}} = a_k + \frac{p_{k-1}}{p_{k-2}}.$$ **[2]**

(ii) Use part (i) to prove that, for each $1 \leqslant k \leqslant n$,

$$\frac{p_k}{p_{k-1}} = [a_k; a_{k-1}, \ldots, a_1, a_0].$$ **[6]**

Question 3 [20 marks].

(a) You are given that

$$\sqrt{53} = [7; \overline{3, 1, 1, 3, 14}].$$

Find all solutions in positive integers x, y to the following equation

$$x^2 - 53y^2 = -1.$$

Explain why you have found ALL solutions. **[8]**

(b) The integer 10037 is a prime number. Given that $3271^2 + 1 \equiv 0 \pmod{10037}$, use Hermite's algorithm to find integers x, y such that

$$x^2 + y^2 = 10037.$$

Explain your working. **[6]**

(c) Let d be an integer such that $d \equiv 3 \pmod 4$. Prove that $x^2 - dy^2 = -1$ has no solutions in positive integers x, y. **[6]**

Question 4 [20 marks]. Let ϕ denote Euler's ϕ-function.

(a) Define **Euler's ϕ-function**. **[3]**

(b) Given a positive integer n let $\mathbb{Z}_n$ denote the ring of integers modulo n. Determine the number of units in $\mathbb{Z}_{315}$. **[4]**

(c) Determine all integers n such that $\phi(n) = 12$. Explain how you know you have found all such integers. **[7]**

(d) Let m and n be positive integers. Prove that $\phi(m)\phi(n) \leqslant \phi(mn)$. **[6]**

Question 5 [12 marks].

(a) For each of the two equations below, determine whether there exists a solution x, y in positive integers by means other than an exhaustive search. If a solution exists explain why. If no solution exists explain why not. Explicitly state any results from the lectures that you use. **[6]**

(i) $x^2 + y^2 = 5850$;

(ii) $x^2 + y^2 = 9450$.

(b) Calculate the value of the Legendre symbol $\left(\frac{99}{101}\right)$ by means other than an exhaustive search. Clearly state the rules that you use for calculating the Legendre symbol. **[6]**

End of Paper.