Queen Mary
University of London

Main Examination period 2019

MTH6127: Metric Spaces and Topology

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

This paper has two sections.
You should attempt all the questions in Section A. In Section B you may attempt as many questions as you wish. Except for the award of a bare pass, only the best TWO questions answered in Section B will be counted.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: R. Buzano, H. Nguyen

In this examination $\mathbb{R}$ stands for the set of real numbers, $\mathbb{Q}$ stands for the set of rational numbers, and $\mathbb{N} := \{1, 2, 3, \ldots\}$ stands for the set of natural numbers.

Section A

You should attempt both questions in this section.

Question 1. [20 marks] Let V be a real vector space.

(a) Define what is meant by a **norm** $\|\cdot\|$ on V and moreover what is meant by a **scalar product** $\langle \cdot\,, \cdot \rangle$ on V. [**5**]

(b) Prove that every scalar product on V induces a norm on V. (You are allowed to assume that every scalar product satisfies the Cauchy-Schwarz inequality $\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle$, as shown in the lectures.) [**5**]

(c) Prove that any norm on V that is induced by a scalar product as in part (b) satisfies the parallelogram law

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \qquad \forall u, v \in V.$$ [**5**]

(d) Let $V = C^0([0, 1])$ denote the vector space of continuous real-valued functions on $[0, 1]$. Prove that the norm

$$\|f\|_{L^1} := \int_0^1 |f(x)|\, dx$$

is **not** induced by a scalar product. [**5**]

Question 2. [20 marks] Let X denote a set.

(a) Define what is meant by a **topology** τ on X. [**3**]

(b) Define what is meant by a **compact** subset of the topological space (X, τ). [**3**]

For the rest of this question, we consider $X = \mathbb{R}$.

(c) Prove that the collection of sets

$$\tau_1 := \{A \subseteq \mathbb{R} \ : \ A = \emptyset \text{ or } A^c \text{ is a countable set}\}$$

defines a topology on $\mathbb{R}$. Here $A^c = \mathbb{R} \setminus A$ is the complement of A. (You may use without proof that the union of finitely many countable sets is countable.) [**7**]

(d) For τ_1 as in part (c), prove that $[0, 1]$ is **not** compact in $(\mathbb{R}, \tau_1)$. [**7**]

Section B

You may attempt as many questions as you wish in this section. Except for the award of a bare pass, only the best TWO questions answered in this section will be counted.

Question 3. [30 marks] Let (X, τ_X) and (Y, τ_Y) be two topological spaces and let $h : X \to Y$ be a function between them.

(a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ in X to **converge** to $x \in X$. **[3]**

(b) Let $X = \mathcal{B}((0,1))$ be the set of bounded real-valued functions on $(0,1)$ with the metric $d(f,g) := \sup_{x\in(0,1)}|f(x) - g(x)|$ and its induced topology $\tau_X = \tau_d$. For $n \in \mathbb{N}$, set

$$f_n(x) = \frac{x^n}{1+n} \qquad \text{and} \qquad g_n(x) = \frac{\cos(\frac{2\pi}{x})}{1+nx}.$$

(i) Prove that the sequence $(f_n)_{n=1}^{\infty}$ converges in (X, τ_X). **[7]**

(ii) Prove that the sequence $(g_n)_{n=1}^{\infty}$ does **not** converge in (X, τ_X). **[7]**

(c) Define what it means for the function $h : X \to Y$ to be **sequentially continuous**. **[3]**

(d) Let $S \subseteq Y$. Define what it means for S to be **sequentially open**. **[3]**

(e) Prove that if $h : X \to Y$ is sequentially continuous and $S \subseteq Y$ is sequentially open, then $h^{-1}(S) \subseteq X$ is sequentially open. **[7]**

Question 4. [30 marks] Let X be a set.

(a) Define what it means for a topology τ on X to be **metrisable**. **[3]**

(b) Prove that if X is finite, only the discrete topology $\tau = \mathcal{P}(X)$ is metrisable. **[7]**

(c) Define what it means for a topology τ on X to be **Hausdorff**. **[3]**

(d) Let (X, τ) be a Hausdorff topological space. Prove:

$$\text{For all } x \in X : \bigcap\{F \subseteq X \;:\; x \in F, F^c \in \tau\} = \{x\}. \qquad (\star)$$ **[7]**

(e) Find an example of a topological space (X, τ) which also satisfies $(\star)$ but which is **not** Hausdorff. Prove that your example indeed has the required properties. **[10]**

Question 5. [30 marks] Let (X, d) be a metric space.

(a) Define what is meant by an **open ball** $B_r(x)$ in (X, d) and moreover what is meant by an **open set** $\Omega \subseteq X$. **[3]**

(b) Let S be a subset of (X, d). Define what is meant by the **interior** of S (denoted $int(S)$) and by the **closure** of S (denoted $cl(S)$). **[4]**

(c) Let T be a subset of (X, d). Define what it means for T to be **connected**. **[3]**

For the rest of this question, consider $X = \mathbb{R}$ with standard metric $d(x, y) := |x - y|$.

(d) Show directly from the definition, not using any other result from the lectures, that $(5, \infty) \setminus \mathbb{N}$ is open in $(\mathbb{R}, d)$. **[5]**

(e) Let

$$A := \{0\} \cup \big([1, 2) \setminus \mathbb{Q}\big) \cup \big((5, \infty) \setminus \mathbb{N}\big).$$

Without justification, find the following sets

$$B := int(A), \qquad C := cl(int(A)), \qquad D := int(cl(int(A))),$$
$$E := cl(A), \qquad F := int(cl(A)), \qquad G := cl(int(cl(A))).$$

(*Hint: All of the seven sets A, B, C, D, E, F, and G are different.*) **[10]**

(f) Which of the seven sets A, B, C, D, E, F, and G from part (e) are connected? Justify your answer. (You may use any result from the lectures provided you make it clear what you are using.) **[5]**

End of Paper.